

# CONSTRUCTIONS OF 3-COLORABLE CORES

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**ABSTRACT.** A finite graph  $G$  is a **core** if every endomorphism of  $G$  is an automorphism. The only 2-colorable core is  $K_2$ . Let  $G$  be a core with chromatic number at least 3. We construct an operation on  $G$  which yields a 3-colorable subdivision  $C(G)$  of  $G$  where  $C(G)$  is also a core, and may have arbitrarily large girth. In addition, there is a homomorphism from  $C(G)$  to  $G$ , in which every vertex of  $G$  is covered by two vertices of  $C(G)$ , and every edge of  $G$  is covered by three edges of  $C(G)$ . Thus, for every core  $G \neq K_2$ , there is a 3-colorable core  $C(G)$  such that (a)  $C(G)$  maps homomorphically onto  $G$ , (b)  $C(G)$  is a topological subdivision of  $G$ , (c)  $G$  is a minor of  $C(G)$ .

Let  $\chi_c(G)$  be the circular chromatic number of  $G$ . Graphs which are  $\chi_c$ -critical are cores. We show that  $G$  is  $\chi_c$ -critical if and only if  $C(G)$  is  $\chi_c$ -critical.

## 1. INTRODUCTION

The inspiration for graph homomorphisms comes originally from topology. Let  $G, H$  be finite, simple graphs, with vertex sets  $V(G), V(H)$  and edge sets  $E(G), E(H)$ . A graph homomorphism from  $G$  to  $H$  is a map from the vertex set of  $G$  to the vertex set of  $H$ , say  $\phi : V(G) \rightarrow V(H)$ , such that whenever two vertices  $u, v$  of  $G$  are connected by an edge in  $G$ , then  $\phi(u), \phi(v)$  are connected by an edge in  $H$ . We often write this as  $G \rightarrow H$ . An excellent beginning survey of graph homomorphisms appears in [HT]; see [N] for a more advanced survey.

The **core** of a graph  $G$  is defined to be its smallest subgraph  $H$  such that  $G \rightarrow H$ . Every finite graph has a unique core up to isomorphism; in this paper we consider only finite graphs. A graph  $G$  is said to be a core if the core of  $G$  is  $G$ . Hence,  $G$  is a core if the only subgraph  $H$  such that  $G \rightarrow H$  is  $G$  itself. Alternatively, a finite graph is a core if every endomorphism of  $G$  is an automorphism.

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Another common name for a core is a retract-rigid graph, since a retract in topology is a subspace  $A$  of space  $B$  for which there is a topological map from  $B$  to  $A$  that fixes  $A$ . Let  $H$  be the core of  $G$  where  $\phi : G \rightarrow H$ . Since  $H$  is a subgraph of  $G$ , there is an inclusion map  $i : H \rightarrow G$ . The restriction of  $\phi$  to  $H$ , say  $\psi$ , is a map from  $H$  to  $H$ . Since  $H$  is the core of  $G$ , it must be its own core, so  $\psi$  is an automorphism of  $H$ . Thus the map  $\phi' = \psi^{(-1)} \circ \phi : G \rightarrow H$  is a homomorphism from  $G$  to  $H$  whose restriction to  $H$  is the identity. For the basic properties of cores, see Hell and Nešetřil [HN]. Imrich and Klavžar [IK] show that the finite product of finitely many triangle-free graphs is a core if and only if every factor is a core. See [B] for results on infinite directed graphs.

There is a strong connection between graph coloring and graph homomorphism. A  $k$ -coloring of a graph  $G$  is an assignment of colors from  $\{1, 2, 3, \dots, k\}$  to the vertices of  $G$  so that whenever two vertices are connected by an edge, they receive different colors. Let  $K_n$  be the complete graph on  $n$  vertices, that is, the graph where every pair of vertices is connected. Then a graph is  $k$ -colorable if and only if  $G \rightarrow K_n$ . We can think of homomorphisms to a fixed graph  $T$  as a generalization of graph coloring. See, for example, [BJH, HNZ, L, NR, NRS, Z2].

Let  $\chi(G)$  be the chromatic number of  $G$ , that is, the smallest integer  $k$  such that  $G$  has a  $k$ -coloring. If  $G \rightarrow H$ , then  $\chi(G) \leq \chi(H)$ . A graph  $G$  is said to be  $\chi$ -critical if all of the vertex-induced proper subgraphs of  $G$  have chromatic number less than  $\chi(G)$ . Thus,  $G \not\rightarrow K_{\chi(G)-1}$ , but  $G - \{v\} \rightarrow K_{\chi(G)-1}$  for any vertex  $v$  in  $V(G)$ . Thus, any  $\chi$ -critical graph is a core. Three simple examples of such cores are the complete graphs, the odd cycles, and the odd wheels.

A generalization of the chromatic number is the circular chromatic number, or star chromatic number. Let  $G_k^t$ , with  $t \leq k/2$ , be the abelian Cayley graph on the vertex set  $\{0, 1, 2, \dots, k-1\}$  where  $i, j$  are connected if  $|i-j| \in \{t, t+1, t+2, \dots, k-t\}$ . Then circular chromatic number of  $G$ , called  $\chi_c(G)$ , is equal to  $\inf\{\frac{k}{t} \mid G \rightarrow G_k^t\}$ . Bondy and Hell [BH] have shown that the infimum is, in fact, a minimum. See also Zhu's alternative definition, [Z3] and his survey, [Z1]. A graph  $G$  is said to be  $\chi_c$ -critical if any of the vertex-induced subgraphs of  $G$  have circular chromatic number less than  $\chi_c(G)$ . Thus, once again, if  $G$  is  $\chi_c$ -critical, then  $G$  is a core.

Since a 2-chromatic graph  $G$  is a core if and only if  $G \cong K_2$ , a natural question to ask is what are the 3-chromatic cores. In this paper, we present a graph operation which transforms any core  $G \not\cong K_2$  into a 3-chromatic graph  $C(G)$ . Moreover, we show that the graph  $C(G)$  has the following properties:

1.  $C(G)$  is a 3-chromatic core,
2.  $C(G)$  maps homomorphically onto  $G$ ,
3.  $C(G)$  is a topological subdivision of  $G$ , hence
4.  $G$  is a minor of  $C(G)$ , and
5.  $G$  is  $\chi_c$ -critical if and only if  $C(G)$  is  $\chi_c$ -critical.

The smallest core of  $G$  is the natural representative of  $G$  in the lattice of equivalence classes of graphs under homomorphism, where  $G$  and  $H$  are equivalent if  $G \rightarrow H$  and  $H \rightarrow G$ . These equivalence classes form a lattice under the partial order given by the relation  $G < H$  if  $G \rightarrow H$ . Thus, our main result shows that when we are interested in the topological embedding properties of cores in this lattice, we need only consider cores with chromatic number less than or equal to 3.

## 2. THE CONSTRUCTION

Let  $C(G)$  be  $G$  with each edge replaced by a path with 3 edges. Let  $D(G)$  be the graph on the same vertex set as  $G$ , where  $u \sim v$  in  $D(G)$  if there is a homomorphism from the path with 3 edges to  $G$  such that one end of the path is vertex  $u$  and the other is vertex  $v$ , that is, there is a path  $u \sim w_1 \sim w_2 \sim v$  in  $G$ , where  $w_2$  may equal  $u$  and  $w_1$  may equal  $v$ , if  $u \sim v$  in  $G$ . These definitions are a special case of the ones given in the proof of Lemma 1 in Hell's and Nešetřil's landmark paper [HN2].

**Lemma 2.1** (Hell and Nešetřil).  $C(G) \rightarrow H$  iff  $G \rightarrow D(H)$ .

In the full lemma, the edges of  $G$  may be replaced by any graph  $I$  with fixed vertices  $i$  and  $j$  such that there is an automorphism of  $I$  which takes  $i$  to  $j$ .

Now  $C(G)$  preserves cycles of  $G$ , but multiplies their length by 3. Thus odd cycles remain odd and even cycles remain even. Hence if  $G$  is bipartite, then  $C(G)$  will also be bipartite. If  $G$  is  $k$ -chromatic, where  $k \geq 3$ , then  $C(G)$  will be 3-chromatic. We see this by observing that we can color all the vertices that correspond to vertices of  $G$  with 1, then color the new vertices, in pairs, with 2 and 3.

For each vertex  $v$  of  $G$  let the corresponding vertex in  $C(G)$  be  $c(v)$ . Label the 2 vertices between  $c(v)$  and  $c(u)$  as  $c(v, u)$  and  $c(u, v)$  where  $c(v, u)$  is adjacent to  $c(v)$  and  $c(u, v)$ ; and  $c(u, v)$  is adjacent to  $c(u)$  and  $c(v, u)$ . We define the natural homomorphism  $\phi : C(G) \rightarrow G$  by

$$(2.1) \quad \phi(w) = \begin{cases} u, & \text{if } w = c(u); \\ v, & \text{if } w = c(v); \\ u, & \text{if } w = c(v, u); \\ v, & \text{if } w = c(u, v) \end{cases}$$

For any two adjacent vertices  $u, v$  in  $G$ , this map identifies  $c(u, v)$  and  $c(v)$ , and also identifies  $c(v, u)$  and  $c(u)$ . Thus any two adjacent vertices in  $C(G)$  have adjacent images under  $\phi$  in  $G$ .

**Lemma 2.2.**  $G \rightarrow H$  iff  $C(G) \rightarrow C(H)$ .

*Proof.* In the first case, suppose that  $G \rightarrow H$ . Then it is straightforward to show that  $C(G) \rightarrow C(H)$ .

Conversely, suppose that  $f : C(G) \rightarrow C(H)$ . Let  $u$  be a vertex in  $V(G)$  and define  $g : G \rightarrow H$  by  $g(u) = \phi(f(c(u)))$ . We will show that  $g$  is a graph homomorphism. Clearly  $g$  is a map from the vertices of  $G$  to the vertices of  $H$ . Therefore, we want to show that if  $u \sim v$  in  $G$ , then  $g(u) \sim g(v)$  in  $H$ . Now  $f$  and  $\phi$  are both graph homomorphisms, but  $c(u) \not\sim c(v)$  in  $C(G)$ ; instead in  $C(G)$  we have the path  $c(u) \sim c(u, v) \sim c(v, u) \sim c(v)$ . Consider what happens when  $f$  acts on this path.

Case 1  $f$  does not act injectively on the path. Then  $f$  cannot identify  $c(u)$  and  $c(v)$ , because  $C(H)$  contains no triangles. Therefore any identification done by  $f$  causes  $f(c(u))$  and  $f(c(v))$  to be adjacent. Since  $\phi$  is a homomorphism,  $\phi(f(c(u))) \sim \phi(f(c(v)))$ .

Case 2  $f$  acts injectively on the path, so that the image under  $f$  of this path with 3 edges is a path with 3 edges in  $C(H)$ , namely  $f(c(u)) \sim f(c(u, v)) \sim f(c(v, u)) \sim f(c(v))$ . Now any path of length 3 in  $C(H)$  must contain at least one vertex corresponding to a vertex in  $H$ , and can contain at most two such vertices.

If the path in  $C(H)$  contains two vertices corresponding to vertices in  $H$ , they must be the two end vertices,  $f(c(u))$  and  $f(c(v))$  because the distance between any two such vertices is at least 3 in  $C(H)$ . Thus,  $\phi$  identifies  $f(c(u))$  and  $f(c(v, u))$ ; since  $f(c(v, u))$  is adjacent to  $f(c(v))$ , and  $\phi$  is a homomorphism, we have that  $\phi(f(c(u)))$  is adjacent to  $\phi(f(c(v)))$ .

If only one vertex in the path corresponds to a vertex  $w$  in  $H$ , then  $w$  corresponds to either  $f(c(u, v))$  or  $f(c(v, u))$ . Without loss of generality, suppose  $w = f(c(u, v))$ . Then  $\phi$  identifies  $f(c(u, v))$  and  $f(c(v))$ , and since  $f(c(u, v))$  is adjacent to  $f(c(u))$ , we have that  $\phi(f(c(u)))$  is adjacent to  $\phi(f(c(v)))$ .

□

We point out that Lemma 2.1 and Lemma 2.2 are not the same statement in disguise. In Lemma 2.1, the graphs  $H$  and  $D(H)$  have the same number of vertices, but in Lemma 2.2,  $C(H)$  has 3 times as many vertices as  $H$ .

**Theorem 2.3.** *Let  $G$  be a connected graph on 3 or more vertices. Then  $G$  is a core if and only if  $C(G)$  is a core.*

*Proof.* Suppose first that  $C(G)$  is a core. If  $G \rightarrow X$  where  $X$  is a proper subgraph of  $G$ , then by Theorem 2.1,  $C(G) \rightarrow C(X)$  and clearly  $C(X)$  is a proper subgraph of  $C(G)$ . This contradicts the fact that  $C(G)$  is a core, so  $G$  must be a core.

Conversely, suppose that  $G$  is a core. Let  $f : C(G) \rightarrow Y$  where  $Y$  is a proper subgraph of  $C(G)$ . We may assume that  $f$  fixes  $Y$  in  $C(G)$  and that  $Y$  is the core of  $C(G)$ . Let  $X$  be the induced graph in  $G$  on  $\{u \in G \mid f(c(u)) \in Y\}$ . We show that  $X$  must be a proper subgraph of  $G$ , and that  $C(G) \rightarrow C(X)$ , hence  $G \rightarrow X$  by Theorem 2.1.

First, if  $Y$  is a proper subgraph of  $C(G)$ , then  $Y$  must be missing some vertex  $w \in C(G)$ . If  $w = c(u)$  for some  $u \in G$ , then  $X$  is also missing  $u$ , so  $X$  is a proper subgraph of  $G$ . Suppose that  $X$  contains  $c(u)$  for all  $u \in G$ , so that  $f$  fixes  $c(u)$  for all  $u \in G$ . Now for any  $u \sim v$ ,  $f$  must act injectively on the path  $c(u) \sim c(u, v) \sim c(v, u) \sim c(v)$ , because any identification will mean that  $c(u) \sim c(v)$ , and these are not adjacent in  $Y$ . Since  $G$  is simple, there is only one path from  $c(u)$  to  $c(v)$  with 3 edges in  $C(G)$  and hence in  $Y$ , so  $f$  fixes  $c(u, v)$  and  $c(v, u)$  as well as  $c(u)$  and  $c(v)$ . Thus if  $Y$  contains  $c(v)$  for every  $v$  in  $G$ , then  $Y \cong C(G)$ .

Next we show that  $C(G) \rightarrow C(X)$ , hence  $G \rightarrow X$ . We have seen that if  $u \sim v$  in  $G$  and  $Y$  contains both  $c(u)$  and  $c(v)$ , then  $Y$  contains  $c(u, v)$  and  $c(v, u)$  as well. If, in contrast, for  $u \sim v$  in  $G$ ,  $Y$  contains  $c(u)$  but not  $c(v)$ , then  $c(v, u)$  can be identified with  $c(u)$ , and  $c(u, v)$  can be identified with any neighbor of  $c(u)$ . If  $c(u)$  has no neighbors in  $Y$ , then  $Y$  is not connected, but if  $G$  is connected, then  $C(G)$  is connected, so its core  $Y$  must also be connected. Thus, since  $Y$  is the smallest subgraph that  $C(G)$  maps to,  $Y$  must be  $C(X)$ . □

It is worthwhile to observe that the same proofs for Lemma 2.2 and Theorem 2.3 will work if all edges of  $G$  are replaced by paths of any fixed odd length.

**Theorem 2.4.** *Let  $G$  be a connected graph, and  $C_i(G)$  be  $G$  with each edge replaced by a path with  $2i + 1$  edges. Then  $G$  is a core if and only if  $C_i(G)$  is a core.*

3.  $\chi_c$ -CRITICALITY

In this section we show that if  $G$  is  $\chi_c$ -critical, then  $C(G)$  is  $\chi_c$ -critical. It is not true that if  $G$  is  $\chi$ -critical, then  $C(G)$  is also  $\chi$ -critical. Take, for example, the 5-wheel,  $W_5$ . We know  $C(W_5)$  is 3-colorable, and contains, but does not equal, an odd cycle, which are the only 3-critical graphs. Thus  $C(G)$  is  $\chi$ -critical only if  $G$  is an odd cycle.

Recall that  $G_k^t$ , with  $t \leq k/2$ , is the abelian Cayley graph on the vertex set  $\{0, 1, 2, \dots, k-1\}$  where  $i, j$  are connected if  $|i-j| \in \{t, t+1, t+2, \dots, k-t\}$ . The circular chromatic number of  $G$ ,  $\chi_c(G)$ , is equal to  $\min\{\frac{k}{t} \mid G \rightarrow G_k^t\}$ . It is a well-known fact that  $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ . Thus, if  $G \not\cong K_2$  is a core, then  $2 < \chi_c(C(G)) \leq 3$ .

**Lemma 3.1.** *Let  $3t > k$ . Then  $D(G_k^t) \cong G_k^{3t-k}$ .*

*Proof.* Let the vertices of  $D(G_k^t)$  be  $\{0, 1, 2, \dots, (k-1)\}$ . Suppose that  $i \sim j$  in  $D(G_k^t)$ ; this is equivalent to the fact that there exists a path  $i \sim w_1 \sim w_2 \sim j$  in  $G_k^t$ . For fixed  $i$ , the set of vertices  $j$  such that  $j$  can be reached from  $i$  by a path with three edges is the set  $\{i+3t, i+3t+1, i+3t+2, \dots, i+3k-3t\}$ , reduced modulo  $k$ . Without loss of generality, since  $G_k^t$  is vertex-transitive, we may choose  $i$  to be 0. Now  $3t > k > 2t$ , hence  $2k > 3t > k$ , so  $0 < 3t-k < k$  and  $0 < 2k-3t < k$ . Thus the neighbors of 0 in  $D(G_k^t)$  are  $\{3t-k, 3t-k+1, \dots, 2k-3t\}$ ; hence  $D(G_k^t) \cong G_k^{3t-k}$ .  $\square$

**Theorem 3.2.** *Let  $3t > k$ . Then  $C(G) \rightarrow G_k^t$  iff  $G \rightarrow G_k^{3t-k}$ .*

*Proof.* Apply Lemma 2.1 and Lemma 3.1.  $\square$

**Corollary 3.3.** *Let  $3t > k$ . Then  $\chi_c(C(G)) = \frac{k}{t}$  if and only if  $\chi_c(G) = \frac{k}{(3t-k)}$ .*

*Proof.* This follows immediately from Theorem 3.2 and the definition of the circular chromatic number.  $\square$

**Theorem 3.4.**  *$G$  is  $\chi_c$ -critical if and only if  $C(G)$  is  $\chi_c$ -critical.*

*Proof.* First, in order to apply Corollary 3.3, we prove that  $\chi_c(C(G)) < 3$  for any  $G$ . Let  $\chi_c(G) = \frac{a}{b}$ , hence  $G \rightarrow G_b^a$ . It is easy to check that  $G_b^a \rightarrow G_{3b}^{3a}$  by the map  $i$  in  $G_b^a$  goes to  $3i$  in  $G_{3b}^{3a}$ . Now by Theorem 3.2, since  $3(a+b) > 3b$ ,  $C(G) \rightarrow G_{3b}^{a+b}$ . Hence  $\chi_c(C(G)) \leq \frac{3b}{a+b} < 3$ .

Secondly, the following numerical fact follows from algebraic manipulation of the ratios. Let  $a, b, c, d$  be positive integers. Then

$$\frac{a}{b} < \frac{c}{d} \text{ if and only if } \frac{a}{3b-a} < \frac{c}{3d-c} .$$

The theorem follows from Corollary 3.3 and the observation that if  $e$  is an edge in  $G$  and  $f_1, f_2, f_3$  are the edges in  $C(G)$  that replaced  $e$ , then

$$C(G) - f_i \longleftrightarrow C(G - e), \text{ for } i = 1, 2, 3.$$

□

These results generalize to the case when all edges of  $G$  are replaced by paths of any fixed odd length, say  $2i + 1$ . Then  $\chi_c(C_i(G)) = \frac{k}{t}$  if and only if  $\chi_c(G) = \frac{k}{(2i+1)t-ik}$ .

**Corollary 3.5.** *Let  $G$  be a connected graph, and  $C_i(G)$  be  $G$  with each edge replaced by a path with  $2i + 1$  edges. Then  $C_i(G)$  is  $\chi_c$ -critical if and only if  $G$  is  $\chi_c$ -critical.*

#### 4. CONTRASTING CONSTRUCTIONS

Our construction  $C(G)$  is not the only way to create a subdivision of a graph which is a core. In  $C(G)$ , we replace every edge by a path of fixed odd length. In Theorem 4.1 we show that we can create a core by replacing all the edges in the outside edge orbit of an odd wheel by a path of odd length. If these edges are replaced by paths of varying lengths, or even lengths, the result is not a core. Also, if we replace the spoke edges of the odd wheel by a path of odd length, we sometimes get a core, and sometimes do not. In particular, Figure 1(a) is a core, but Figure 1(b) is bipartite and hence not a core; Figure 1(c) is a core, but Figure 1(d) is not a core.

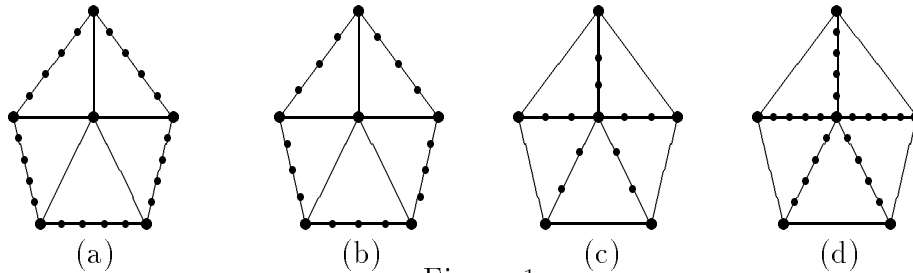


Figure 1

Define  $W(s, m)$  as a cycle  $C$  with  $s(1 + m)$  vertices, labeled as  $\{1, 2, 3, \dots, s(1 + m)\}$ , plus an extra center vertex  $v$  which is connected to  $s$  evenly spaced vertices on  $C$ , say  $\{1, 2 + m, 1 + 2(1 + m), \dots, 1 + (s - 1)(1 + m)\}$ . Thus  $W(s, m)$  has  $s(1 + m) + 1$  vertices and  $s(2 + m)$  edges. The large cycle is odd when  $s$  is odd and  $m$  is even; the small cycles from the center vertex to consecutive neighbors on  $C$  are odd when  $m$  is even and even when  $m$  is odd. Figure 1(a) shows  $W(5, 4)$  and Figure 1(b) is  $W(5, 3)$ .

**Theorem 4.1.**  *$W(s, m)$  is a core if and only if  $s$  is odd and  $m$  is even. Further, any graph  $G$  which is a cycle  $C$  plus an extra vertex  $v$  is a core if and only if  $G \cong W(2k + 1, 2j)$  for some positive integers  $k$  and  $j$ .*

*Proof.* Let  $G = W(s, m)$ . Suppose that  $m$  is odd. Then  $C$  is even, and the small cycles are even, so  $G$  is bipartite, and not a core.

Suppose that  $m$  is even and  $s$  is even. Then  $C$  is even and the small cycles are odd. In this case,  $G$  maps to one of the small cycles. Then the set  $\{v, 1, 2, 3, \dots, 1 + m, 1 + (1 + m)\}$  is a small cycle. We map the neighbors of  $v$  to 1 and  $1 + (1 + m)$ , and let the small cycles follow in the natural way. Send  $1 + j(m + 1)$  to 1 if  $j$  is even and to  $1 + (1 + m)$  if  $j$  is odd, for  $2 \leq j \leq s - 1$ .

Suppose that  $m$  is even and  $s$  is odd. Then  $C$  is odd, and the small cycles are odd. The girth of  $G$  is the same as the size of a small cycle. In any map of  $G$  to itself, the center vertex  $v$  cannot be identified with any vertex on the cycle, because the resulting graph has smaller girth than  $G$ . Thus the vertices in  $C$  must map within  $C$ , but  $C$  is an odd cycle and hence a core itself. Therefore  $G$  is a core.

Now suppose that  $G$  is a cycle  $C$  plus an extra vertex  $v$ . Let the neighbors of  $v$  in consecutive order relative to  $C$  be  $w_1, w_2, \dots, w_t$ , which are not evenly spaced. Then  $w_2 \sim v \sim w_1$  and the path from  $w_1$  to  $w_2$  is a cycle. If this cycle is even, we can map it to the path  $w_1 \sim v \sim w_2$ . Therefore, assume that all small cycles in  $G$  are odd. Suppose that the small cycle formed by  $v$  and  $w_1, w_2$  and the path from  $w_1$  to  $w_2$  is a smallest cycle of  $G$ , such that the adjacent small cycle composed of  $w_3 \sim v \sim w_2$  and the path from  $w_2$  to  $w_3$  is not a smallest cycle of  $G$ . Since both cycles are odd, the larger one has at least 2 more vertices than the smaller. Let the vertices in the path from  $w_1$  to  $w_2$ , excluding  $w_1, w_2$ , be  $x_1 \sim x_2 \sim \dots \sim x_m$  and the vertices in the path from  $w_2$  to  $w_3$ , excluding  $w_2, w_3$ , be  $y_1 \sim y_2 \sim \dots \sim y_n$ . We fix  $w_3$  and map  $y_n$  to the center vertex  $v$ ,  $y_{n-1}$  to  $w_1$ , and the rest of the path from  $y_{n-2}$  to  $y_1$  onto the path from  $x_1$  to  $x_m$ , ending with  $y_1$  maps to  $x_m$ . Since  $n$  is odd and  $n \geq m + 2$ , the path from  $y_{n-2}$  to  $y_1$  has the same parity and at least as many vertices as the path from  $x_1$  to  $x_m$ . Since  $x_m$  is adjacent to  $w_2$ , so is the image of  $y_1$ . Thus  $G$  maps to a subgraph of itself, and is not a core.  $\square$

Note that it matters which edges we choose to replace by paths. If we replace only the spoke edges of  $W_5$  by paths of length 2, shown in Figure 1(c), the resulting graph is a core, because the outside 5-cycle is the unique smallest cycle of the graph and hence remains fixed, and the center vertex cannot be identified with any vertex in that cycle



without making a triangle. If we replace the spoke edges of  $W_5$  by paths of length 4, Figure 1(d), however, the resulting graph maps to the 5-cycle, by sending the center vertex to a vertex in the outside cycle and wrapping all the resulting 5-cycles around the outside 5-cycle. Thus this graph is not a core.

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