

Dependent Edges in Mycielski Graphs and 4-Colorings of 4-Skeletons

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Abstract

A dependent edge in an acyclic orientation of a graph is one whose reversal creates a directed cycle. In answer to a question of Erdős, Fisher et al. [5] define $d_{min}(G)$ to be the minimum number of dependent edges of a graph G , where the minimum is taken over all acyclic orientations of G , and also $r_{m,k}$ as the supremum of the ratio $d_{min}(G)/e(G)$, where $e(G)$ is the number of edges in G and the supremum is taken over all graphs G with chromatic number m and girth k . They show that $r_{m,k} \leq \frac{(m-2)}{m}$ and $r_{4,4} \geq \frac{1}{20}$. We show that $r_{5,4} \geq \frac{4}{71}$, $r_{6,4} \geq \frac{7}{236}$, and $r_{7,4} \geq \frac{11}{735}$ and that the Mycielski construction on a triangle-free graph with at least one dependent edge yields a graph with at least 3 dependent edges. In addition, we give an alternative proof of the answer to Erdős's question, based on Tysdal [6] and Youngs [7].

1 Introduction

In every acyclic orientation of a triangle, there is one edge that can be reversed to make the triangle a directed cycle. Define a dependent edge in an acyclic orientation of a graph to be one whose reversal creates a directed cycle. Paul Erdős asked [1] if every graph with girth at least 4 can be given an acyclic orientation with no dependent edge. Such a graph would be a cover graph, *i.e.*, the Hässe diagram of a partially ordered set. There are infinitely many graphs for which the answer is yes. It is well known that if the chromatic number of graph G is strictly less than its girth, then G has an orientation with no dependent edges (see, for example, [4]). The argument goes as follows: color the vertices of G with colors $\{1, 2, 3, \dots, \chi(G)\}$, and orient each edge from its lower-numbered vertex to its higher-numbered vertex. It is easy to check that this is an acyclic orientation, and, since the girth of G is greater than the chromatic number, every cycle of G will have at least two edges in each direction.

The answer in general, however, is no. There are infinitely many graphs for which every acyclic orientation has at least one dependent edge. In 1997 Fisher, et. al. [5] gave a proof that the Grötzsch graph (see Figure 1) always has at least one dependent edge in every acyclic orientation. (In some papers, the Grötzsch graph is also called the Mycielski graph, see, for example, [3].) Thus, every graph that contains the Grötzsch graph as a subgraph always has at least one dependent edge in every acyclic orientation.

Tysdal [6] recently observed that a 1996 proof by Youngs [7] of the fact that every non-bipartite graph that can be quadrilaterally embedded on the projective plane cannot be 3-colorable implies that every such graph has a dependent edge in every acyclic orientation. The Grötzsch graph is one such example, see Figure 2 for its embedding on the projective plane. In section 2, we give the proof that the Grötzsch graph always has at least one dependent edge using the ideas of Youngs' proof. In addition, Tysdal has shown that when applied to any odd cycle, the Mycielski construction (defined precisely in Section 2), yields a graph in which every acyclic orientation has a dependent edge. These graphs all have chromatic number 4 and girth 4, while quadrilateral embeddings on the projective plane that are not bipartite have chromatic number bounded between 4 and 6, and girth either 3 or 4.

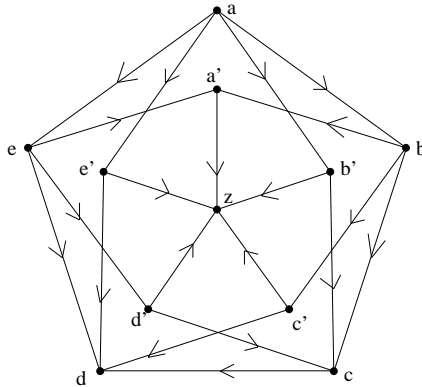


Figure 1: An Acyclic Orientation of the Grötzsch Graph with One Dependent Edge

Given that there are some graphs for which every acyclic orientation has dependent edges, Fisher et al. ask what is the minimum percentage of dependent edges that a given graph can have in any acyclic orientation. They suggested looking at the supremum of this ratio, where the supremum is taken over the class of graphs having fixed chromatic number and girth. If $d_{\min}(G)$ is the minimum number of dependent edges of a graph G , where the minimum is taken over all acyclic orientations of G , and $r_{m,k}$ is the supremum of the ratio $d_{\min}(G)/e(G)$, where the supremum is taken over all graphs G with chromatic number m and girth k , then by the argument above, $r_{j,k} = 0$ if

$j < k$. Figure 1 shows an acyclic orientation of M , the Grötzsch graph, with exactly one dependent edge. In this figure, the edge (e, d) is dependent in the 4-cycle $edcd'$. Since [5] showed that $d_{\min}(M) \geq 1$, this orientation yields $d_{\min}(M) = 1$, and hence $r_{4,4} \geq 1/20$. Whether or not $r_{4,4} = 1/20$ is still open.

Fisher et al. also showed that for all m and k , $r_{m,k} \leq (m-2)/m$. In this paper we prove that $r_{5,4} \geq \frac{4}{71}$, $r_{6,4} \geq \frac{7}{236}$, and $r_{7,4} \geq \frac{11}{755}$ by analyzing iterations of the Mycielski construction. We also show that if we start with a triangle-free graph G with the property that every acyclic orientation has at least 1 dependent edge, then the Mycielski construction on G yields a graph for which every acyclic orientation has at least 3 dependent edges.

2 Graphs on the Projective Plane

Recall the following construction of Mycielski. Graph definitions and terminology will follow [2]. Given a graph G , construct a new graph $M(G)$ in the following way. For each vertex $v \in V(G)$, add a corresponding vertex v' , and make v' adjacent to $u \in V(G)$ only if v is adjacent to u in G . (Thus the neighbors of a new vertex are the neighbors in G of the original vertex to which it corresponds.) Now add an extra vertex z , and connect it to each of the other new vertices. This graph will have the following useful properties: the chromatic number of $M(G)$ will be one more than the chromatic number of G , and if G has no triangles, then neither does $M(G)$. Also, if G is connected, then so is $M(G)$.

Thus $M(C_5)$ is constructed by starting with a pentagon, adding five new vertices, each of whom gets two neighbors from the original pentagon, and then adding an extra vertex adjacent to the new five.

Theorem 2.1. *Every acyclic orientation of the Mycielskian $M(C_5)$ has at least one dependent edge.*

Proof. The idea for this proof comes from Youngs [7]. This graph contains ten 4-cycles. If it is possible to give this graph an acyclic orientation with no dependent edges, then none of the 4-cycles will be directed, and switching the direction of any edge will not cause any of the 4-cycles to become directed. In other words, each 4-cycle must have two edges oriented clockwise and two edges oriented counter-clockwise.

We can embed this graph on the projective plane in order to get a better view of what is happening with the 4-cycles. See Figure 2. Vertices a through e appear twice in our drawing due to the nature of the projective plane, but are really identical. The other vertices (those on the “inside” of our drawing) appear only once.

All ten 4-cycles are now faces, and all the faces are 4-cycles.

Suppose it is possible to give this graph an acyclic orientation with no dependent edges. Given such an orientation, let us give each edge within a 4-cycle a +1 or -1 value: +1 if the edge is oriented clockwise within that 4-cycle, and -1 if the edge is oriented counter-clockwise within the 4-cycle. Now each 4-cycle can be given a numerical value, namely the sum of the values of its edges. The possible values of a 4-cycle are -4, -2, 0,

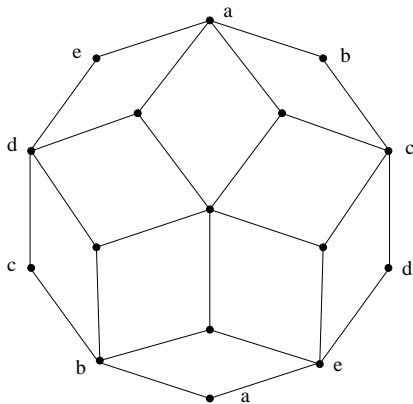


Figure 2: The Grötzsh Graph Viewed on the Projective Plane

2 and 4. Because of the orientation that we gave the graph, we know that each 4-cycle must have two edges going in each direction. In other words, the value of each 4-cycle must be zero. We will now show that this cannot happen.

If the value of each 4-cycle were zero, then we would have $\sum N(F) = 0$, where F is a 4-cycle, $N(*)$ is the value of the cycle $*$, and the sum is taken over all 4-cycles in the graph. Let us now consider how each edge in the graph contributes to this sum. Each of the edges appears in two 4-cycles, and hence gets counted twice.

First consider the edges that appear once in our drawing (those on the *inside*). Each of these edges gets counted once as going clockwise and once as going counter-clockwise. Thus each such edge contributes zero to the sum.

Edges that appear twice in our drawing (those on the *outside*), get counted both times in the same direction. Thus these edges each contribute either a +2 or a -2. But there are five of these edges, and so there is no way to make their contributions add to zero. Thus the sum of the values of the 4-cycles is not zero, so there is some 4-cycle whose value is not zero, meaning that at least three of its edges are going the same direction. \square

Corollary 2.2. *Every non-bipartite graph that has a quadrilateral embedding on the projective plane has a dependent edge in every acyclic orientation.*

Proof. Let G be a non-bipartite graph that is quadrilaterally embedded on the projective plane. Since G is not bipartite, G has an odd cycle. Cut the projective plane so that G has the odd cycle on the boundary of the projective plane. The above argument shows that G has a dependent edge. \square

A graph G that is non-bipartite and has a quadrilateral embedding on the projective plane is called a 4-skeleton. Youngs [7] showed that a 4-skeleton cannot be 3-colorable,

and in addition, there must be a quadrilateral face on which all four colors appear. We observe the following corollary to his theorem. Note that there are three non-isomorphic ways to use four colors on a 4-cycle.

Corollary 2.3. *Let G be a 4-skeleton. In every 4-coloring of G there are at least three 4-cycles that use all four colors, and three of these 4-cycles have non-isomorphic colorings.*

Proof. Given a 4-coloring of G , there are $4! = 24$ possible orderings for the four colors. Each ordering of the colors specifies an acyclic orientation by orienting each edge toward the endpoint with later color in the ordering. By the argument in Theorem 2.1, for every acyclic orientation there is a 4-cycle in G with a dependent edge. A dependent edge arises in a 4-cycle only when its vertices have the four colors appearing in increasing order (according to the ordering of the colors) around the cycle. There are three non-isomorphic ways to color a 4-cycle with four colors. Thus every 4-coloring of G must have at least one 4-cycle colored in each of these ways. In other words, a 4-coloring of G must have at least three 4-cycles colored with all four colors, and three of these colorings must be non-isomorphic □

If the fourth color is used exactly once, then it must be in each of the three 4-cycles. Tysdal [6] has shown that every (proper) 4-coloring of $M(C_5)$ where one color is used only once yields an acyclic orientation with exactly one dependent edge [6].

3 Lower Bounds on $r_{m,k}$

In pursuit of lower bounds for $r_{m,4}$, we have considered graphs obtained by applying iterations of the Mycielski construction. Our first theorem below applies to every graph with girth at least 4 and with at least one dependent edge. Such a graph would have chromatic number at least 4, and $M(C_5)$ is the smallest such graph [3].

We use the notation $G - a + b$ to denote the subgraph of $M(G)$ induced by the replacement of a with b in $V(G)$.

Theorem 3.1. *If G is a triangle-free graph such that $d_{\min}(G) \geq 1$, then $d_{\min}(M(G)) \geq 3$.*

Proof. Since G is a subgraph of $M(G)$, every acyclic orientation of $M(G)$ has at least one dependent edge. Fix an acyclic orientation of $M(G)$. Let $e_1 \in E(G)$ be dependent (in this orientation), and say $e_1 = (x_1, y_1)$, where $x_1, y_1 \in V(G)$. Note that it is undetermined which vertex is the head of the oriented edge. Let x'_1 be the new vertex corresponding to x_1 . Let $G' = G - x_1 + x'_1$. This is a copy of G in $M(G)$ that does not contain e_1 , and yet it must have a dependent edge. So $M(G)$ contains a second dependent edge $e_2 = (x_2, y_2)$, which appears in G' . Again we do not specify which vertex is the head of the directed edge e_2 .

Case 1: The edge e_2 is not incident to x'_1 , *i.e.*, $x'_1 \notin \{x_2, y_2\}$. Since G does not contain a triangle, x_1 cannot be adjacent to both x_2 and y_2 . Suppose, without loss of generality, that x_1 and x_2 are not adjacent, and x'_2 is the new vertex corresponding to x_2 . Let $G'' = G - x_1 - x_2 + x'_1 + x'_2$. Now G'' is a copy of G in $M(G)$ that does not contain e_1 or e_2 , hence there must be a third dependent edge in $M(G)$.

Case 2: The edge e_2 is incident to x'_1 , *i.e.*, without loss of generality, $x'_1 = x_2$. Let y'_1 be the new vertex corresponding to y_1 and let $G''' = G - y_1 + y'_1$. Now G''' is a copy of G in $M(G)$ that does not contain e_1 or e_2 (since y_1 and $x_2 = x'_1$ are missing). Since the orientation of G''' must have a dependent edge, again $M(G)$ has a third dependent edge. \square

Using the ideas from the proof above, it is easy to show that if G is triangle-free and $d_{\min}(G) \geq 2$, then $d_{\min}(M(G)) \geq 4$. Similarly, if G is triangle-free and $d_{\min}(G) \geq 3$, then $d_{\min}(M(G)) \geq 6$.

By Theorem 3.1, $d_{\min}(M^2(C_5)) \geq 3$, and hence $r_{5,4} \geq 3/71$. We can improve this bound by looking at particulars of $M(C_5)$, and we show below that $4 \leq d_{\min}(M^2(C_5)) \leq 7$. We find lower bounds for $d_{\min}(M^3(C_5))$ and $d_{\min}(M^4(C_5))$.

We will use the following vertex labelling for $M^2(C_5)$. In this labelling, vertices a, b, c, d and e represent the vertices of the original 5-cycle. When $M(C_5)$ is created, the vertices a', b', c', d' and e' are copies of the original vertices, and z_1 is the extra vertex added at the end of the construction. When the Mycielski construction is applied again, the notation is the following: x''' is a copy of x , x'' is a copy of x' , z_2 is the copy of z_1 , and z_3 is the extra vertex added.

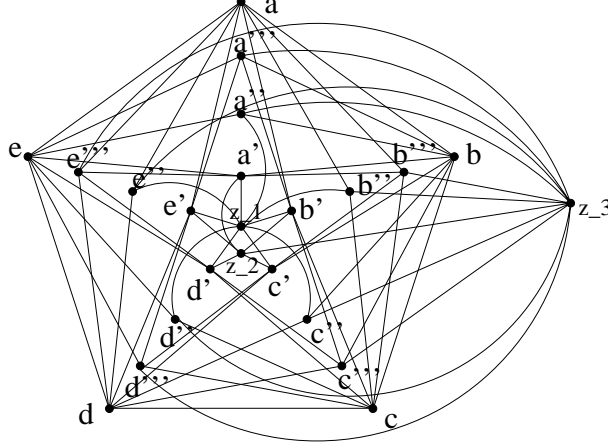


Figure 3: Vertex Labelling of $M^2(C_5)$

Lemma 3.2. *Given any two vertices in $M^2(C_5)$, neither of which are z_3 , there exists a copy of $M(C_5)$ in $M^2(C_5)$ that misses both vertices.*

Proof. If neither vertex is z_1 or z_2 , then see Figure 4. If the two vertices are any of z_1, z_2 , a vertex with label type x' , or a vertex with label type y'' , then see Figure 5. If one vertex is z_1 or z_2 and the other is x (a vertex of the original 5-cycle), then see Figure 6. If one vertex is z_1 or z_2 and the other is a vertex of label type x''' , then see Figure 7.

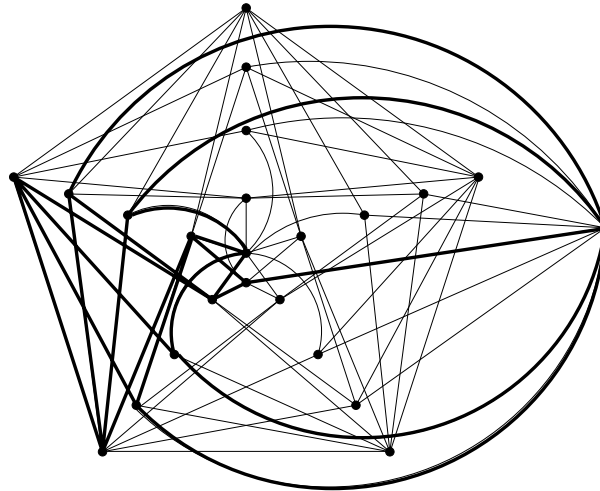


Figure 4: A copy of $M(C_5)$ in $M^2(C_5)$ with z_1, z_2

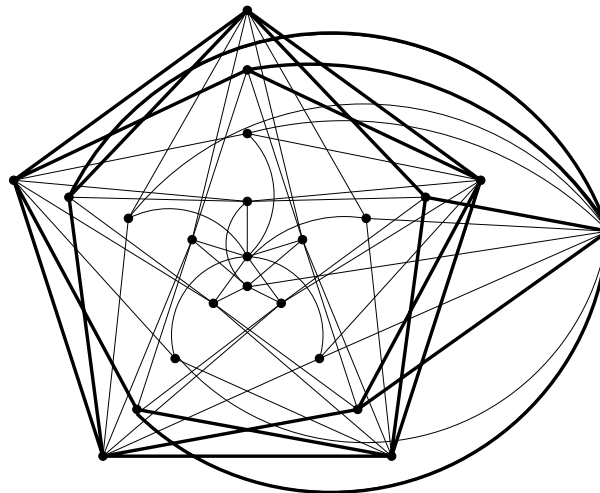


Figure 5: A copy of $M(C_5)$ in $M^2(C_5)$ with none of z_1, z_2, x' or y''

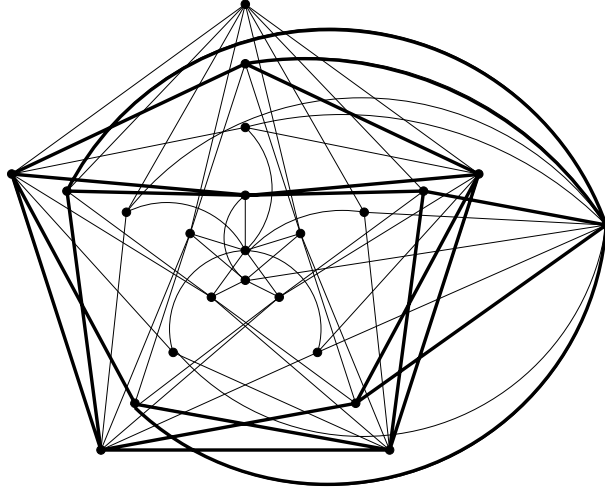


Figure 6: A copy of $M(C_5)$ in $M^2(C_5)$ with none of z_1, z_2, x

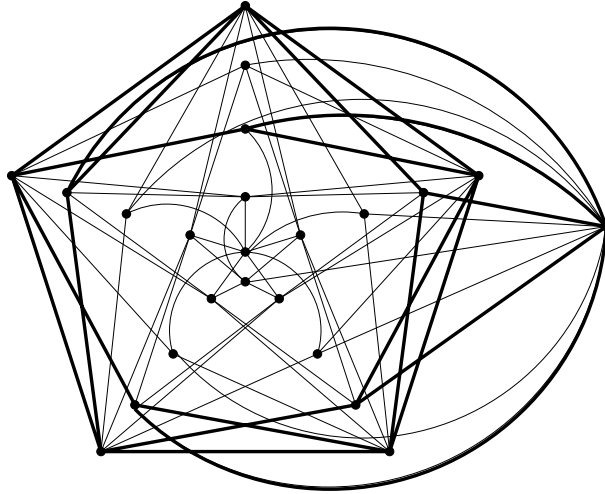


Figure 7: A copy of $M(C_5)$ in $M^2(C_5)$ with none of z_1, z_2, x'''

□

Theorem 3.3. $M^2(C_5)$ has at least four dependent edges in every acyclic orientation.

Proof. The following proof is based on [6]. We will show that no matter how the three edges found in the proof of Theorem 3.1 are arranged, there is a copy of $M(C_5)$ in $M^2(C_5)$ that misses all three edges, thus implying the existence of another dependent edge. We may assume that none of the three edges are incident to z_3 , by the construction

of the three dependent edges in Theorem 3.1. Further, if the three dependent edges already found in the preceding proof are covered by two vertices, we can use Lemma 3.2, to find a copy of $M(C_5)$ in $M^2(C_5)$ that misses both vertices, and thus get a fourth dependent edge. Thus we assume that the three dependent edges from the preceding proof cannot be covered by two vertices and the subgraph induced by the three edges must have six vertices. Let this set of six vertices be S .

Case 1: Suppose that both z_1 and z_2 are in S . They are not adjacent, so they cover two edges. If the third edge has an endpoint of the form x or x'' , Figure 6 will give us a copy of $M(C_5)$ that misses the three known dependent edges, and if the third edge has an endpoint of the form x' or x''' , then Figure 7 will do the same.

Case 2: Neither z_1 nor z_2 is in S . We will find a copy of $M(C_5)$ in $M^2(C_5)$ symmetric to Figure 4 that does not contain any of the three known dependent edges. Erase the primes from the vertices in S and consider the subgraph formed by these unprimed vertices and the images of the three dependent edges in the original C_5 , $\{a, b, c, d, e\}$. This subgraph has at most three edges. Any of the edges not in the subgraph generate a copy of $M(C_5)$ symmetric to Figure 4.

Case 3: One of z_1 or z_2 is in S . We will find a copy of $M(C_5)$ in $M^2(C_5)$ symmetric to Figure 4 that does not contain any of the three known dependent edges. Except for z_1 or z_2 , erase the primes from the vertices in S and consider the subgraph formed by these unprimed vertices and the images of the two dependent edges which are not adjacent to z_1 or z_2 in the original C_5 , $\{a, b, c, d, e\}$. We want to find an edge in the original C_5 which is not in the subgraph, and is not incident to the unprimed vertex which was originally in a dependent edge with z_1 or z_2 . Such an edge must exist, since there are three edges not in the subgraph, and any vertex in the C_5 can be incident to at most two edges. This edge generates a copy of $M(C_5)$ symmetric to Figure 4 which does not contain any of the three dependent edges. \square

Corollary 3.4. $r_{5,4} \geq \frac{4}{71}$.

While we know that $d_{\min}(M^2(C_5)) \geq 4$, we have been unable to show equality. Our best orientation, shown in Figure 8, has seven dependent edges, shown in boldface. This orientation arises from the given 5-coloring of $M^2(C_5)$. Simply orient each edge toward the vertex with the larger number. The proof that these are the only dependent edges is by exhaustion.

Next we use the fact that $d_{\min}(M^2(C_5)) \geq 4$ to find lower bounds for $d_{\min}(M^3(C_5))$ and $d_{\min}(M^4(C_5))$. An independent set of vertices in a graph G is a subset of $V(G)$ in which no two vertices are adjacent.

Theorem 3.5. *Every acyclic orientation of $M^3(C_5)$ has at least seven dependent edges.*

Proof. Fix an acyclic orientation of $M^3(C_5)$. By construction, $M^3(C_5)$ is an original copy of $M^2(C_5)$, say A , with an independent set of vertices, say B , where each vertex in B corresponds to one vertex in A , plus one more vertex. By Theorem 3.3, we know that every copy of $M^2(C_5)$ in $M^3(C_5)$ has at least four dependent edges, and in particular

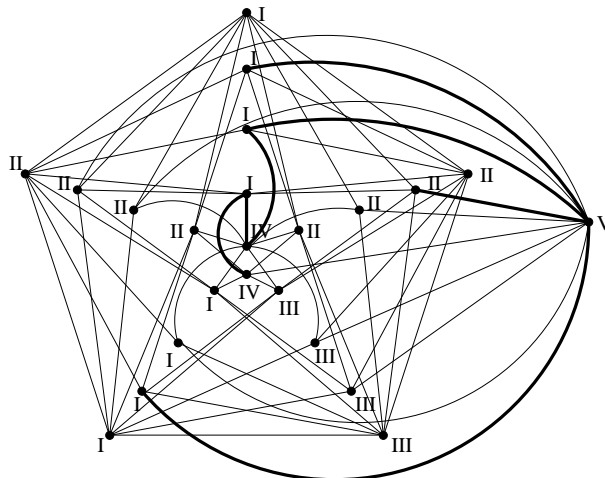


Figure 8: $M^2(C_5)$ with Seven Dependent Edges

A does. Let H be the subgraph of A induced by the vertices of 4 dependent edges in A . We claim that we can find an independent set in H that covers at least 3 of these dependent edges. We then switch this independent set with their corresponding vertices in B , and obtain a new copy of $M^2(C_5)$ which has at most 1 of the 4 original dependent edges. Since this new copy must have at least 4 dependent edges, we gain an additional 3 dependent edges, for a total of at least 7 dependent edges in $M^3(C_5)$.

We observe that if H has an independent set of size 3, since every vertex is adjacent to at least one dependent edge, this independent set covers at least 3 dependent edges. Since $M^4(C_5)$ has no triangles, the neighbors of any vertex of degree 3 in H form an independent set of size 3. Hence we assume that the degree of every vertex in H is less than or equal to 2. Thus, the connected components of H are paths and cycles. If H is bipartite, it contains an independent set which covers all 4 dependent edges. If H is not bipartite, it contains a chordless odd cycle; if this odd cycle has size 7 or larger, H has an independent set of size 3. If the odd cycle has size 5, and H has any other vertex not on the 5-cycle, then H has an independent set of size 3. If H is exactly a 5-cycle, then we can choose two independent vertices to cover any set of 4 edges, hence we can find an independent set that covers all 4 dependent edges.

□

Corollary 3.6. $r_{6,4} \geq \frac{7}{236}$.

It is possible to find a copy of the Moebius ladder on 8 vertices (an 8-cycle with chords between each pair of vertices opposite vertices) as a subgraph of $M^2(C_5)$. This graph has largest independent set of size 3, and has perfect matching. Thus we cannot immediately strengthen the argument above to get a better bound for $M^3(C_5)$. We can, however, use this result to give a bound for the number of dependent edges in $M^4(C_5)$.

Theorem 3.7. $M^4(C_5)$ has at least eleven dependent edges.

Proof. The method of proof is similar to the proof of Theorem 3.5. Fix an acyclic orientation of $M^4(C_5)$. Let A be the original copy of $M^3(C_5)$ in $M^4(C_5)$ and let H be the subgraph of A induced by the vertices of seven of its dependent edges. Since $M^4(C_5)$ has no triangles, clearly H is not a complete graph. We claim that we can find an independent set in H that covers at least four of the dependent edges. We can switch these vertices with their corresponding vertices in $M^4(C_5)$, getting a new copy of $M^3(C_5)$ which does not contain at least 4 of the original 7 dependent edges. Thus we find an additional 4 dependent edges in $M^4(C_5)$ for a total of 11.

We observe that if H has an independent set of size 4, since every vertex is adjacent to at least one dependent edge, this independent set covers at least 4 dependent edges.

Case 1: There is an independent set $\{u, v\}$ in H such that every other vertex in H is adjacent to one or both of u and v . If $\{u, v\}$ covers 4 dependent edges, we choose $\{u, v\}$; if not, then $\{u, v\}$ covers at most 3 dependent edges, but since H contains 7 dependent edges, there are 4 which are not incident to either u or v . These edges must go between neighbors of u and neighbors of v . Hence the set of neighbors of u in H , which must be independent, covers these 4 dependent edges.

Case 2: H has an independent set of size 3, say $\{u, v, w\}$. If $\{u, v, w\}$ covers 4 dependent edges, we choose $\{u, v, w\}$. If not, then each of u, v, w must be incident to at least one dependent edge, so $\{u, v, w\}$ covers exactly 3 dependent edges. For any vertex x , let $N(x)$ be the neighbors of x in H . The vertices of $H - \{u, v, w\}$ are the union (not necessarily disjoint) of $N(u), N(v)$ and $N(w)$. There are 4 dependent edges in $H - \{u, v, w\}$. By the pigeon-hole principle, one of $N(u), N(v), N(w)$ covers at least 3 of these 4 dependent edges, say $N(u)$ does. Then $N(u)$ also covers the unique dependent edge incident to u , so $N(u)$ covers at least 4 dependent edges. \square

Corollary 3.8. $r_{7,4} \geq \frac{11}{755}$.

Clearly the bounds for $r_{5,4}$, $r_{6,4}$ and $r_{7,4}$ could be improved simply by finding the exact value for $d_{\min}(M^2(C_5))$. It may also be possible to improve them by looking at other graphs within the appropriate classes. In general, we observe that if G is any graph such that $d_{\min}(G) > 0$, then $d_{\min}(M(G)) \geq d_{\min}(G) + 1$. This follows because we can always trade at least one vertex of G for a new vertex in $M(G)$. If $d_{\min}(G) = 0$, then $d_{\min}(M(G))$ may not increase, for instance if G is an edge, then $M(G)$ is the 5-cycle and both can be oriented with no dependent edges.

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