RETRACTS OF ODD-ANGULATED GRAPHS AND CONSTRUCTION OF CORES

ZHONGYUAN CHE AND KAREN L. COLLINS

Abstract. Let G be a connected graph with odd girth $2k+1$. Then G is a $(2k+1)$-angulated graph if every two vertices of G are joined by a path such that each edge of the path is in some $(2k+1)$-cycle. We prove that if G is $(2k+1)$-angulated, and H is connected with odd girth at least $2k+3$, then any retract R of the box (or Cartesian) product $G \Box H$ is isomorphic to $S \Box T$ where S is a retract of G and T is a subgraph of H. A graph G is strongly $(2k+1)$-angulated if any two vertices of G are connected by a sequence of $(2k+1)$-cycles with consecutive cycles sharing at least one edge. We prove that if G is strongly $(2k+1)$-angulated, and H is connected with odd girth at least $2k+1$, then any retract R of the Cartesian (or box product) $G \Box H$ is isomorphic to $S \Box T$ where S is a retract of G and T is a subgraph of H or S is a single vertex and T is a retract of H. These two results improve theorems on weakly and strongly triangulated graphs by Nowakowski and Rival in [9]. As a corollary, we get that the core of two strongly $(2k+1)$-angulated cores must be either one of the factors or the product itself. We construct cores from graphs that have a vertex which is fixed under any of its automorphisms, and also from vertex-transitive graphs. In particular, the box product $M(G) \Box M(G)$ is a core if $M(G)$ is a core, where $M(G)$ is the graph resulting from the Mycielski construction on G. Further, the box product of any two Kneser graphs $K(n, 2n+1) \Box K(m, 2m+1)$ is a core whenever $n, m \geq 2$; and $K(n, 2n+1) \Box C_{2m+1}$ is a core for $m \geq n \geq 2$.

1. Introduction

Graphs in this paper will be simple, loopless and finite unless otherwise specified. We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. If two vertices u, v are adjacent, we write $u \sim v$. We will denote $(2k+1)$-cycles by C_{2k+1} and the complete graph with n vertices by K_n.

A graph homomorphism between graphs G and H is a map $f : V(G) \to V(H)$, or $G \to_f H$, that preserves adjacency, *i.e.,* if $f : G \to H$, then $u \sim v$ in G implies that $f(u) \sim f(v)$ in H. A graph H is called a retract of G if H is an induced subgraph of G and there is a graph homomorphism $f : G \to H$. The map f is called a retraction of G. A smallest retract of G is known as the core of G, and we denote it by G^*. It is easy to see that G is its own core if and only if every endomorphism of G is an automorphism of G. Complete graphs, odd cycles, Kneser graphs and (vertex or edge) chromatic-critical graphs are all cores.

The Cartesian product, or box product, $G \Box H$, is defined to be the graph that has vertex set $V(G) \times V(H)$ and $(g_1, h_1) \sim (g_2, h_2)$ in $G \Box H$ if either $g_1 = g_2$ and $h_1 \sim h_2$ in H or $g_1 \sim g_2$ in G and $h_1 = h_2$. Let U be a subgraph of $G \Box H$. The projection of U to G is $\{a \mid a$ is a first coordinate of some vertex in $U\}$.

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Let G be a connected graph. Then G is called weakly triangulated if each edge of G is in a triangle; G is called strongly triangulated if every pair of vertices is joined by a sequence of triangles with consecutive triangles sharing an edge. These definitions, and decomposition theorems for retracts of the box product graph $G\Box H$ are derived in [9] when one factor is weakly-triangulated, or when one factor is strongly-triangulated. Some incorrect proofs are fixed in [7].

In this paper, we replace the definition of weakly triangulated by the more general definition of $(2k+1)$-angulated, $k \geq 1$, and extend the theorems of [9] to graphs with larger odd girth. Let G be a connected graph with odd girth $2k+1$, $k \geq 1$. Then we define G to be $(2k+1)$-angulated if every two vertices of G are joined by a path such that each edge of the path is in some $(2k+1)$-cycle; and we define G to be strongly $(2k+1)$-angulated if any two vertices of G are connected by a sequence of $(2k+1)$-cycles with consecutive cycles sharing at least one edge. Our definition of $(2k+1)$-angulated improves their definition of weakly triangulated because a weakly triangulated graph is always 3-angulated, but a 3-angulated graph is a weakly triangulated graph, plus some (possibly empty) set of edges which are not in triangles. In addition, although a weakly triangulated graph need not be a strongly triangulated graph, and vice versa, it is easy to see that strongly $(2k+1)$-angulated graphs are also $(2k+1)$-angulated.

Our results generalize the weakly triangulated and strongly triangulated theorems in [9] from $k=1$ to any positive integer k. Lemma 2.2 is a direct generalization, but while the main ideas of our proofs of Lemmas 3.1, 3.2, 3.3, and Theorems 3.4, 3.5 and 4.1 are similar, Nowakowski's and Rival's proofs depend on the isometry between a graph and its retract while our proofs do not.

In Section 2, we prove the basic lemma that transfers smallest odd cycles throughout a box product of graphs. In Section 3, we prove the strongest result possible for $(2k+1)$-angulated graphs, namely that if G is $(2k+1)$-angulated and H is a connected graph with odd girth at least $2k+3$, then any retract of $G\Box H$ is $S\Box T$ where S is a retract of G and T is a connected subgraph of H. On the other hand, if G and H are both $(2k+1)$-angulated cores, then the core of $G\Box H$ need not be one of G, H, or $G\Box H$, see [3].

In Section 4, we prove that if G is strongly $(2k+1)$-angulated, and the odd girth of H is at least $2k+1$, then any retract of $G\Box H$ is $S\Box T$ where either S is a retract of G and T is a subgraph of H, or $|V(S)| = 1$ and T is a retract of H. Hence we show that if G and H are strongly $(2k+1)$-angulated cores, then the core of $G\Box H$ is either G, H or $G\Box H$. Let $M(G)$ be the result of the Mycielski construction on G. We show that if G is a 5-angulated graph which is chromatic-critical, then $M(G)\Box M(G)$ is a core. With a more general version of the Mycielski construction, this result generalizes for chromatic-critical $(2k+1)$-angulated graphs.

In Section 5, we consider the box product of two vertex-transitive graphs with large odd girth. In particular, we show the box product of two Kneser graphs $K(n, 2n+1)\Box K(m, 2m+1)$ is a core for any integers $m, n \geq 2$, as well as the box product $K(n, 2n+1)\Box C_{2m+1}$ where $m \geq n \geq 2$. We conclude with some open questions in Section 6.

2. Retracts of Box Products

In this section, we prove a basic lemma about where smallest odd cycles of G can be mapped by a retraction of $G\Box H$. Let R denote a retract of the box product $G\Box H$

and let $\phi : G\Box H \to R$ be the retraction from $G\Box H$ to R. We know that R can be considered as an induced subgraph of $G\Box H$. For $x \in V(H)$, let $R_x = (G\Box\{x\}) \cap R$. As observed in [9], if G is connected, then any retract R of G is connected, and R_x is connected for any vertex x in H.

Definition 2.1. [9] Let G and H be finite connected graphs and S be a connected subgraph of G. We say that S **transfers** in H if for every retract R of $G\Box H$, whenever $x, y \in V(H)$, and R_y is not empty, and $S\Box\{x\} \subseteq R$, then $S\Box\{y\} \subseteq R$.

A set $C = \{u_1, u_2, \cdots, u_{2k+1}\} \subseteq V(G)$ is an odd cycle of G if $u_1 \sim u_2 \sim \cdots \sim u_{2k+1} \sim u_1$ and there are no other edges between the vertices in C.

Lemma 2.2. Let G and H be finite connected graphs, where the odd girth of G is $2k+1$, and let R be any retract of $G\Box H$. Let C be an odd cycle in G of length $2k+1$. If there exists $x, y \in H$ and $u_1 \in C$ such that $C\Box\{x\} \subseteq R$ and $(u_1, y) \in R$, then $C\Box\{y\}$ is in R.

Proof. It is enough to show that $C\Box\{y\}$ is in R for some y adjacent to x in H since H is connected. Let $C = \{u_1, u_2, \cdots, u_{2k+1}\}$. Since $\phi((u_2, y))$ is adjacent to (u_1, y) and (u_2, x), and both are in R, either $\phi((u_2, y)) = (u_2, y) \in R$ or $\phi((u_2, y)) = (u_1, x)$. If $\phi((u_2, y)) = (u_2, y)$, that is, (u_2, y) is fixed under ϕ, then either $\phi((u_3, y)) = (u_3, y) \in R$ or $\phi((u_3, y)) = (u_2, x)$. Let us suppose that (u_{j+1}, y) is the first vertex in $C\Box\{y\}$ that is not fixed under ϕ, that is, $\phi((u_i, y)) = (u_i, y)$ for all $1 \le i \le j$, then $\phi((u_{j+1}, y)) = (u_j, x)$ in R. Now $\phi((u_{j+2}, y))$ is adjacent to both $\phi((u_{j+1}, y)) = (u_j, x)$ and (u_{j+2}, x). This implies that $\phi((u_{j+2}, y)) = (w_1, x)$ where w_1 is adjacent to u_j and u_{j+2} in G. Next $\phi((u_{j+3}, y))$ is adjacent to both $\phi((u_{j+2}, y)) = (w_1, x)$ and (u_{j+3}, x). This implies that $\phi((u_{j+3}, y)) = (w_2, x)$ where w_2 is adjacent to w_1 and u_{j+3} in G. Continue this argument, then we can see that $\phi((u_{2k}, y)) = (w_{2k-j-1}, x)$ where w_{2k-j-1} is adjacent to w_{2k-j-2} and u_{2k} in G. Finally, $\phi((u_{2k+1}, y)) = (w_{2k-j}, x)$ since it has to be adjacent to (w_{2k-j-1}, x) and (u_{2k+1}, x). Also $\phi((u_{2k+1}, y))$ must adjacent to $\phi((u_1, y)) = (u_1, y)$, hence $w_{2k-j} = u_1$. Thus,

$$\phi(C\Box\{y\}) = \{u_1, \cdots, u_j\}\Box\{y\} \cup \{u_j, w_1, w_2, \cdots, w_{2k-j-1}, u_1\}\Box\{x\}$$

Hence $\{u_1, \cdots, u_j, w_1, w_2 \cdots, w_{2k-j-1}\}$ contains an odd cycle in G with size at most $2k-1$. This contradicts the fact that the odd girth of G is $2k+1$. Hence $\phi(C\Box\{y\}) = C\Box\{y\}$. □

3. One Factor is $(2k+1)$-angulated

In this section we show that if G is $(2k+1)$-angulated and H has odd girth at least $2k+1$, then $2k+1$ cycles of G transfer in H. Hence $R_x = R_y$ whenever both R_x and R_y are not empty. We conclude with the theorems that if G is $(2k+1)$-angulated and H has odd girth at least $2k+3$, then any retract of $G\Box H$ is the form of $S\Box T$, where S is a retract of G and T is a subgraph of H.

Let G be a connected graph with odd girth $2k+1$ and H be a connected graph with odd girth at least $2k+3$. We note that if C is a $2k+1$ cycle in G, then for any vertex $x \in V(H)$, $\phi(C\Box\{x\}) = C'\Box\{y\}$ for some $2k+1$ cycle $C' \subseteq G$ and some vertex $y \in V(H)$. Further, if both G, H have odd girth $2k+1$, then either $\phi(C\Box\{x\}) = C'\Box\{y\}$ for some $2k+1$ cycle $C' \subseteq G$ and some vertex $y \in V(H)$, or $\phi(C\Box\{x\}) = \{a\}\Box C''$ for some $2k+1$ cycle $C'' \subseteq H$ and some vertex $a \in V(G)$.

Lemma 3.1. Let G be $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+1$, and R be a retract of $G\Box H$. Then R_x is $(2k+1)$-angulated for any $x \in H$.

Proof. Let $\phi : G\Box H \to R$ be the retraction and let (a,x) and (b,x) be any two vertices in R_x. Then there is a path $P := \{a = u_0 \sim u_1 \sim u_2 \sim \cdots u_{n-1} \sim u_n = b\}$ in G joining a and b such that each edge $\{u_{i-1} \sim u_i\}$ is contained in some $(2k+1)$-cycle C_i of G for $1 \le i \le n$ since G is $(2k+1)$-angulated.

If the odd girth of H is at least $2k+3$, then $\phi(C_i\Box\{x\}) = C_i'\Box\{z_i\}$ where C_i' is some $(2k+1)$-cycle in G and $z_i \in V(H)$ for each $1 \le i \le n$. Since $\phi(a,x) = (a,x) \in C_1'\Box\{z_1\}$, then $z_1 = x$, and $\phi(C_1\Box\{x\}) = C_1'\Box\{x\}$. Note that any two consecutive cycles $C_i'\Box\{x\}$ and $C_{i+1}'\Box\{x\}$ overlap on at least one vertex, hence we see that $z_i = x$ and $\phi(C_i\Box\{x\}) = C_i'\Box\{x\}$ for each $1 \le i \le n$. Therefore, the vertices (a,x) and (b,x) are joined by the path $\{a = u_0' \sim u_1' \sim u_2' \sim \cdots \sim u_{n-1}' \sim u_n' = b\}\Box\{x\}$, and each edge $\{(u_{i-1}',x) \sim (u_i',x)\}$ is contained in the $(2k+1)$-cycle $C_i'\Box\{x\}$ in R_x for $1 \le i \le n$. Therefore, R_x is $(2k+1)$-angulated.

If the odd girth of H is $2k+1$, then for each $(2k+1)$-cycle $C_i\Box\{x\}$, either $\phi(C_i\Box\{x\}) = C_i'\Box\{z\}$ where C_i' is an $(2k+1)$-cycle in G and $z \in H$ or $\phi(C_i\Box\{x\}) = \{w\}\Box\{C_i''\}$ where C_i'' is a $(2k+1)$-cycle in H and $w \in G$. Suppose that the projection of $\phi(P\Box\{x\})$ to G is $\{a = w_0 \sim w_1 \sim w_2 \cdots \sim w_s = b\}$. Then for each w_i there exists some j_i such that $(w_i, z_i) = \phi(u_{j_i}, x) \sim \phi(u_{j_i+1}, x) = (w_{i+1}, z_i)$ for each w_i, and $\{(w_i,z_i),(w_{i+1},z_i)\}$ is contained in some $(2k+1)$-cycle $C_{j_i}'\Box\{z_i\}$ in $G\Box\{z_i\}$, $0 \le i \le s-1$. Recall that $a = w_0 \in C_{j_0}'$ and $(a,x) \in R_x$, thus, $\phi(C_{j_0}'\Box\{x\}) \subseteq R_x$ by Lemma 2.2. In addition, every pair of consecutive cycles overlap on at least one vertex. Therefore $C_{j_i}'\Box\{x\} \subseteq R_x$ by Lemma 2.2 for all $0 \le i \le s-1$. Hence the vertices (a,x) and (b,x) are joined by the path $\{a = w_0 \sim w_1 \sim w_2 \cdots \sim w_s = b\}\Box\{x\}$, and each edge $\{(w_i,x) \sim (w_{i+1},x)\}$ in the path is contained in the $(2k+1)$-cycle $C_{j_i}'\Box\{x\}$ in R_x. Therefore, R_x is $(2k+1)$-angulated. □

Example 1: We note that the condition of H having odd girth at least $2k+1$ is necessary, because of the following example. Although C_5 is 5-angulated, and K_3 is connected, we demonstrate a retraction from $C_5\Box K_3$ to $K_2\Box K_3$, and K_2 is not 5-angulated. Let the vertices of $C_5\Box K_3$ be $\{a,b,c,d,e\} \times \{1,2,3\}$. Define $\phi : C_5\Box K_3 \to \{a,b\}\Box\{1,2,3\}$ as follows. The vertices $(a,1),(b,1),(a,2),(b,2),(a,3),(b,3)$ are fixed by ϕ. Let $\phi(c,i) = (b,i+1)$ modulo 3, $\phi(d,i) = (b,i+2)$ modulo 3, and $\phi(e,i) = (b,i)$ modulo 3. Then it is straightforward to check that ϕ is a homomorphism.

Lemma 3.2. Let G be $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+1$. Then every $(2k+1)$-cycle of G transfers in H.

Proof. Let C be an odd cycle of G and let $\phi : G\Box H \to R$ be a retraction. Suppose that $C\Box\{x\} \subseteq R_x$ and there is some vertex $(d,y) \in R_y$. We show that $C\Box\{y\} \subseteq R_y$. If $d \in C$, apply Lemma 2.2. Let $d \notin C$, and choose any vertex $a \in C$. Then there exists a path $P := \{a = u_0 \sim u_1 \sim u_2 \sim \cdots \sim u_{n-1} \sim u_n = d\}$ in G such that each edge $\{u_{i-1} \sim u_i\}$ is contained in some $(2k+1)$-cycle C_i of G for $1 \le i \le n$ (since G is $(2k+1)$-angulated).

If the odd girth of H is at least $2k+3$, then, using the same argument as in Lemma 3.1, for each $1 \le i \le n$, $\phi(C_i\Box\{x\}) = C_i'\Box\{x\}$ where C_i' is some $(2k+1)$-cycle in G. Note that $\phi(d,x) \in \phi(C_n\Box\{x\}) = C_n'\Box\{x\}$, and $\phi(d,x) \sim (d,y)$ in R, therefore, $\phi(d,x) = (d,x)$ and $d \in C'_n$. Recall that $(d,y) \in R_y$, thus, $C'_n \Box \{y\} \subseteq R_y$ by Lemma 2.2. In addition, every two consecutive cycles overlap on at least one common vertex, therefore, $C'_i \Box \{y\} \subseteq R_y$ by Lemma 2.2 for all $n-1 \geq i \geq 1$. Thus, $(a,y) \in R_y$ since $a \in C'_1$. This implies that $C \Box \{y\} \subseteq R_y$ by Lemma 2.2.

Let the odd girth of H be $2k+1$. Then $(d,x) \sim (d,y)$ so $\phi(d,x) = (e,y)$ or $\phi(d,x) = (d,z)$ where $e \sim d$ in G or $y \sim z$ in H. Assume that $\phi(d,x) = (e,y)$. Denote the projection of $\phi(P \Box \{x\})$ to G as $\{a = w_0 \sim w_1 \sim w_2 \cdots \sim w_t = e\}$. For each w_i, there exist some j_i such that $(w_i, z_i) = \phi(u_{j_i}, x) \sim \phi(u_{j_{i+1}}, x) = (w_{i+1}, z_i)$ and the edge $\{(w_i, z_i), (w_{i+1}, z_i)\}$ is contained in some $(2k+1)$-cycle $C'_{j_i} \Box \{z_i\}$ in $G \Box \{z_i\}$, $0 \leq i \leq t-1$. Note that $e \in C'_{j_{t-1}}$ and $(e,y) \in R_y$, hence $C'_{j_{t-1}} \Box \{y\} \subseteq R_y$ by Lemma 2.2. In addition, every two consecutive cycles overlap on at least one vertex, therefore $C'_{j_i} \Box \{y\} \subseteq R_y$ by Lemma 2.2 for all $t-1 \geq i \geq 0$. Thus, $(a,y) \in R_y$ since $a \in C'_{j_0}$. This implies that $C \Box \{y\} \subseteq R_y$ by Lemma 2.2.

Suppose that $\phi(d,x) = (d,z)$, where $z \sim y$ in H. The proof in this case follows the same plan: project $(P \Box \{x\})$ to G and use Lemma 2.2. □

Lemma 3.3. Let G be $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+1$. If R is any retract of $G \Box H$, then R_x and R_y are identical for any two vertices $x, y \in H$ when both R_x, R_y are not empty.

Proof. If G is $(2k+1)$-angulated, then R_x is $(2k+1)$-angulated by Lemma 3.1. If $|R_x| \geq 2$, then there is a path joining any two vertices in R_x where each edge in the path is contained in some $(2k+1)$-cycle in R_x. If R_y is not empty, then each of these cycles transfers to R_y by Lemma 3.2. Thus, if $|R_x| \geq 2$ for at least one vertex $x \in H$, then $|R_y| \geq 2$ or R_y is empty for all $y \in V(H)$. Any cycles in R_y transfer back to R_x, hence $R_x = R_y$. If $|R_x| \leq 1$ for all $x \in V(H)$, then R_x has no edges for any x, hence R_x is either empty or isomorphic to the one vertex graph for each x. □

Theorem 3.4. Let G be $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+1$. If R is a retract of $G \Box H$, then $R = S \Box T$ where where S, T are connected and $S \subseteq G$ and $T \subseteq H$.

Proof. Let R be a retract of $G \Box H$. If R_x and R_y are not empty where $x, y \in V(H)$, then R_x and R_y are identical by lemma 3.3. Hence $R = S \Box T$ where $S = \{a \in V(G) |\ (a,y) \in R_y\}$, $T = \{y \in V(H) | R_y \neq \emptyset\}$. We know S is connected because R_x is connected, and T is connected because H connected. □

Theorem 3.5. Let G be $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+3$. Let $\phi : G \Box H \to R$ be a retraction. Then $R = S \Box T$ where S, T are connected where S is a retract of G and $T \subseteq H$.

Proof. Note that $R = S \Box T$ by Theorem 3.4. Let $x \in T$. If $|R_x| = 1$, then $|R_x| = 1$ for all $x \in T$ by Lemma 3.3. Therefore, $R = T \subseteq H$, but the odd girth of H is greater than the odd girth of G, so $G \Box H$ cannot map to H. Therefore S cannot be a single vertex, and $|R_x| \geq 2$ for all $x \in V(T)$.

Let $(a,x) \in G \Box \{x\}$, but not in $S \Box \{x\}$. For any $(b,x) \in S \Box \{x\}$, there is a path $\{(b,x) \sim (a_1,x) \sim \cdots \sim (a_m,x) = (a,x)\}$ joining (b,x) and (a,x) such that each edge in the path is contained in some $(2k+1)$-cycle of $G \Box \{x\}$ since G is $(2k+1)$-angulated. Say the cycles are $C_1 \Box \{x\}, C_2 \Box \{x\}, \ldots, C_m \Box \{x\}$, where C_i is a $(2k+1)$-cycle of G for $1 \leq i \leq m$. Then $\phi(C_i \Box \{x\}) \subseteq S \Box \{z_i\}$ where each $z_i \in V(H)$ for each $1 \leq i \leq m$. Since $C_1 \Box \{x\}$ overlaps $S \Box \{x\}$ at (b,x), then

$\phi(C_1\Box\{x\}) \subseteq S\Box\{x\}$, and since C_1 overlaps C_2, then $\phi(C_2)\Box\{x\} \subseteq S\Box\{x\}$ as well. As we continue along the path, we see that $\phi(C_m\Box\{x\}) \subseteq S\Box\{x\}$, hence $\phi(a,x) \in S\Box\{x\}$ since $(a,x) \in C_m\Box\{x\}$, so $\phi(G\Box\{x\}) \subseteq S\Box\{x\}$. Hence S is a retract of G. □

Remark: This theorem does not hold if the odd girth of H is $2k+1$, see [3].

Corollary 3.6. Suppose G is a $(2k+1)$-angulated core, with a vertex that is fixed by every automorphism of G, and H is a core with odd girth at least $2k+3$. Then $G\Box H$ is a core.

Proof. Let $\phi : G\Box H \to R$ be a retraction. Then $R = G\Box T$ where $T \subseteq H$ by Theorem 3.5. The odd girth of G is less than the odd girth of H. Therefore, for each $x \in V(H)$, there exists $y_x \in V(T)$ such that $\phi(G\Box\{x\}) = G\Box\{y_x\}$. Let $\psi : H \to T$ by $\psi(x) = y_x$. We will show that ψ is a homomorphism, hence $T = H$ because H is a core. It is clear that ψ is well-defined. Now let $x \sim z$ in H. Then $\phi(G\Box\{x\}) = G\Box\{y_x\}$ and $\phi(G\Box\{z\}) = G\Box\{y_z\}$. Let a be the vertex in G which is fixed by every automorphism. Since $x \sim z$, $(a,x) \sim (a,z)$. Therefore $(a,y_x) = \phi(a,x) \sim \phi(a,z) = (a,y_z)$. Hence $y_x \sim y_z$.

4. One Factor is Strongly $(2k+1)$-angulated

Since strongly $(2k+1)$-angulated implies $(2k+1)$-angulated, all the lemmas from the previous section apply to strongly $(2k+1)$-angulated graphs. In this section we prove our main theorems about strongly $(2k+1)$-angulated graphs. Then we apply the result to construct cores of box product form with one factor having a fixed vertex under any automorphism of it. In particular, suppose $M(G)$ is a Mycielski construction of a graph G. If $M(G)$ is a core, then $M(G)\Box M(G)$ is a core. This result can generalized to the box product of graphs with the construction similar to the Mycielski construction to create 4-chromatic graphs of arbitrarily large odd girth. In Example 1, both C_5 and K_3 are strongly angulated, it has a retract $K_2\Box K_3$, so the hypothesis in the theorem below cannot be relaxed to allow the odd girth of H to be less than $2k+1$.

Theorem 4.1. Let G be strongly $(2k+1)$-angulated and H be a connected graph with odd girth at least $2k+1$. Let $\phi : G\Box H \to R$ be a retraction. Then $R = S\Box T$ where either S is a retract of G or $|V(S)| = 1$ and T is a retract of H.

Proof. If G is strongly $(2k+1)$-angulated, then it is $(2k+1)$-angulated. Therefore, $R = S\Box T$ by Theorem 3.4. If $V(S) = \{a\}$, then $R = \{a\}\Box T \cong T$, hence T is a retract of $G\Box H$. If $|V(S)| \neq 1$, choose two different vertices (a,x) and (b,x) in $S\Box\{x\} \subseteq G\Box\{x\}$. Let $\phi : G\Box H \to S\Box T$ be the retraction. We will show $\phi : G\Box x \to S\Box\{x\}$ for any $x \in V(T)$. For any $(g,x) \in G\Box\{x\}$ that is not in $S\Box\{x\}$, there is a sequence of $(2k+1)$-cycles with consecutive cycles sharing at least one edge joining (a,x) and (g,x) in $G\Box\{x\}$. Let $C\Box\{x\}$ be the cycle that contains (a,x). If $\phi(C\Box\{x\}) \subseteq R_x$, then the image under ϕ of all of the cycles are contained in R_x. In this case, $\phi(g,x) \in S\Box\{x\}$. On the other hand, if $\phi(C\Box\{x\})$ in not contained in R_x, then $\phi(C\Box\{x\}) = \{a\}\Box C'$ for some $2k+1$ cycle C' in H, and therefore, all the cycles are contained in $\{a\}\Box T$. Hence $\phi(g,x) = (a,y)$ for some vertex $y \in V(H)$. The same construction shows that either $\phi(g,x) = (b,z)$ for some vertex $z \in V(H)$ which is impossible because $a \neq b$. Hence $\phi(g,x) \in S\Box\{x\}$ for any $(g,x) \in G\Box\{x\}$. □

Corollary 4.2. Suppose G and H are strongly $(2k+1)$-angulated cores, then the core of $G \square H$ is either G, H or $G \square H$. Further, if G and H are homomorphically inequivalent (that is, $G \not\to H$ and $H \not\to G$), then $G \square H$ is a core.

Recall the Mycielski construction on a graph G with vertex set $\{u_1, u_2, \ldots, u_n\}$: let $M(G)$ be the graph whose vertex set is

$$U = \{u_1, u_2, \ldots, u_n\} \cup W = \{w_1, w_2, \ldots, w_n\} \cup Z = \{z\}$$

The adjacencies in $M(G)$ are as follows: $u_i \sim u_j$ in $M(G)$ if $u_i \sim u_j$ in G, and $w_i \sim u_j$ if $u_i \sim u_j$ in G, and $z \sim w_i$ for $1 \leq i \leq n$. Note that if G is chromatic-critical, then $M(G)$ is also chromatic-critical, hence a core.
Example 2: the Petersen graph [6], P, and the 4-chromatic Mycielski graph [2], or Grötzsch graph, $M(C_5)$, are both strongly 5-angulated cores. It is well-known that $P \square P$ is a core [1], and we give a new proof of this in Section 5. Since $M(C_5)$ is 4-chromatic, but P is 3-chromatic, the core of $P \square M(C_5)$ cannot be P. On the other hand, the core of $P \square M(C_5)$ cannot be $M(C_5)$ because P does not map to $M(C_5)$. To see this, we observe that identifying two vertices of P would create a triangle, but $M(C_5)$ has no triangles, so if P mapped to $M(C_5)$, there would be a 10 vertex subgraph of the 11 vertex graph $M(C_5)$ which contained P. It is straightforward to check that that cannot happen. Hence $P \square M(C_5)$ is a core.

From Corollary 4.2, we can conclude that $M(C_5) \square M(C_5)$ is a core. If not, then the core of $M(C_5) \square M(C_5)$ is $M(C_5)$, but $M(C_5)$ is a core, and has exactly one vertex, u, of degree 5. Hence in any mapping of $M(C_5)$ to itself, u must go to u. Now in $M(C_5) \square M(C_5)$, we can find two adjacent copies of $M(C_5)$, say $M(C_5) \square \{x\}$ and $M(C_5) \square \{y\}$. If $M(C_5) \square M(C_5) \to M(C_5)$, then $(u, x) \sim (u, y)$ both map to the same vertex u, which cannot happen. Hence $M(C_5) \square M(C_5)$ is a core. The same proof shows the following corollary.

Corollary 4.3. Suppose G is a strongly $(2k+1)$-angulated core, such that in any automorphism of G, there is a vertex which remains fixed. Then the core of $G \square G$ is a core.

Lemma 4.4. Let G be a strongly 5-angulated graph. Then $M(G)$ is strongly 5-angulated.

Proof. The vertex set of $M(G)$ is $U \cup W \cup Z$. Any two vertices in U are connected by a sequence of 5-cycles which overlap on at least an edge because U has the same edges as G. If we consider two vertices $w_1, w_2 \in W$, then if $u_1 \sim u_2$, the vertices u_1, u_2, w_1, w_2, z form a 5-cycle. Hence z is in a 5-cycle with every vertex in U and in W. If $u_1 \not\sim u_2$, then the graph on $U - \{u_1, u_2\} + \{w_1, w_2\}$ is isomorphic to G, so there is a sequence of consecutive 5-cycles that overlap on at least one edge between w_1 and w_2. If we consider $u_i \in U$ and $w_j \in W$, then if $i = j$, we already showed these were in a 5-cycle together, and if $i \neq j$, then the graph on $U - \{u_j\} + \{w_j\}$ contains u_i, w_j and is isomorphic to G, so they are connected by a sequence of 5-cycles which overlap on at least an edge. □

Corollary 4.5. Let G be a strongly 5-angulated graph. If $M(G)$ is a core, then $M(G) \square M(G)$ is a core. Hence if G is chromatic-critical, $M(G)$ and $M(G) \square M(G)$ are both cores.

Starting with graphs G of any odd girth, Van Ngoc and Tuza [11] and independently Youngs [12] have used a construction similar to the Mycielski construction to create 4-chromatic graphs of arbitrarily large odd girth. It is straightforward to show that Lemma 4.4 holds in this more general setting, hence so does a version of Corollary 4.4.

These results cannot extend to $G\Box G\Box G$, or longer products, because if G is strongly $(2k+1)$-angulated, then $G\Box G$ is not strongly $(2k+1)$-angulated, although it is $(2k+1)$-angulated. See open question 6.2.

□

5. Constructing Cores using Vertex-Transitive Graphs

In this section, we use the results from section 3 and section 4 to construct cores from vertex-transitive graphs. In particular, we show that $K(n, 2n+1)\Box K(m, 2m+1)$ is a core for any integers $m, n \geq 2$.

Definition 5.1. A graph G is called vertex-transitive if there is an automorphism f of G such that $f(u) = v$ for any two vertices u and v of G.

Definition 5.2. Let $\alpha(G)$ be the size of a maximum independent set of G. The independence ratio of a graph is defined to be $i(G) = \frac{\alpha(G)}{|V(G)|}$.

If $G \to H$ and H is vertex transitive, then $i(G) \geq i(H)$ by the No-Homomorphism Lemma [1]. Hence, if G, H are vertex transitive and homomorhically equivalent, then $i(G) = i(H)$.

Theorem 5.3. Let G and H be strongly $(2k+1)$-angulated cores. If G, H are vertex-transitive and $i(G\Box H) < i(G), i(H)$. Then $G\Box H$ is a core.

Proof. If G and H are strongly $(2k+1)$-angulated, Corollary 4.2 tells us that $(G\Box H)^*$ is G, H or $G\Box H$. Since the core of a vertex-transitive graph is vertex-transitive [5], then $i(G\Box H) = i((G\Box H)^*)$ by the No-Homomorphism lemma [1]. Hence $G\Box H$ is a core. □

Theorem 5.4. Let G and H be strongly $(2k+1)$-angulated cores. Let G be vertex-transitive such that such that any two maximum independent sets have non-empty intersection (that is, $i(G\Box K_2) < i(G)$). If $G \not\to H$, then $G\Box H$ is a core.

Proof. If G and H are strongly $(2k+1)$-angulated, Corollary 4.2 tells us that $(G\Box H)^*$ is G, H or $G\Box H$. Now $G\Box K_2 \not\to G$, because $G\Box K_2$ and G are both vertex-transitive, but $i(G\Box K_2) \neq i(G)$. Therefore $i(G\Box K_2) \not\to G$, hence $G\Box H$ is a core. □

Theorem 5.5. Let G be a $(2k+1)$-angulated vertex-transitive core such that $i(G\Box K_2) < i(G)$. If H is a core with odd girth at least $2k+3$, then $G\Box H$ is a core.

Proof. Let $\phi : G\Box H \to (G\Box H)^*$. By Theorem 3.5, we know that $(G\Box H)^* = G\Box T$ where $T \subseteq H$. Therefore, for each $x \in V(H)$, there exists $y_x \in V(T)$ such that $\phi(G\Box\{x\}) = G\Box\{y_x\}$. Let $\psi : H \to T$ by $\psi(x) = y_x$. We will show that ψ is a homomorphism, hence $T = H$ because H is a core. It is clear that ψ is well-defined. Suppose that $x \sim z$ in H. Then we show that $y_x \neq y_z$. Otherwise, $\phi(G\Box K_2) \cong G$ which contradicts $i(G\Box K_2) < i(G)$. Thus, $y_x \neq y_z$. Let $a \in V(G)$. Then $(a, x) \sim (a, z)$ in $G\Box H$, and $(a', y_x) = \phi(a, x) \sim \phi(a, z) = (a'', y_z)$. Hence $a' = a''$ and $y_x \sim y_z$. Hence ψ is a retraction from H to T, but H is a core, so $T = H$.

□

Definition 5.6. The Kneser graph $K(n,s)$ is the graph whose vertices are the n-subsets of an s-set where $1 \leq n < \frac{s}{2}$, and two vertices are adjacent if and only if the corresponding n-subsets are disjoint. Kneser graphs are vertex transitive. The Petersen graph, P, is $K(2,5)$.

Recall that the odd girth of $K(n,2n+1)$ is $2n+1$. If an odd cycle C_{2i+1} maps to $K(n,2n+1)$, then $i(C_{2i+1}) = \frac{i}{2i+1} \geq \frac{n}{2n+1} = i(K(n,2n+1))$ by the No-homomorphism lemma [1]. Hence $i \geq n$ which means that the odd girth of $K(n,2n+1)$ is at least $2n+1$. S. Stahl showed the existence of a $(2n+1)$-cycle in $K(n,2n+1)$ in [10]. Therefore the odd girth of $K(n,2n+1)$ is $2n+1$. Here we have a much stronger result which implies that $K(n,2n+1)$ is strongly $(2n+1)$-angulated.

Lemma 5.7. Any two vertices $u,v \in V(K(n,2n+1))$ are in some $(2n+1)$-cycle of $K(n,2n+1)$ together.

Proof. We will first find one $(2n+1)$-cycle in $K(n,2n+1)$. Since $K(n,2n+1)$ is vertex-transitive, it is enough to show that for any fixed vertex is in a $(2n+1)$-cycle with any other vertex of $K(n,2n+1)$.

Consider the vertices $u_i = \{(i-1)n+j \mid 1 \leq j \leq n\}$ for $1 \leq i \leq n$ in $K(n,2n+1)$ where addition is mod $2n+1$. Let $1 \leq t \leq n$. Note that the intersection of u_1 and u_{2t} has $t-1$ elements: $u_1 \cap u_{2t} = \{n-(t-2), n-(t-3), \ldots, n\}$. The intersection of u_1 and u_{2t+1} has $n-t$ elements: $u_1 \cap u_{2t+1} = \{1,2,\ldots,n-t\}$. Thus, the case $t=1$ shows $u_1 \sim u_2$ and the case $t=n$ shows $u_1 \sim u_{2n+1}$, but for every other value of t, $u_1 \not\sim u_{2t}, u_{2t+1}$. Similarly, u_i is adjacent to u_{i+1} and u_{2n+2-i} for $2 \leq i \leq n+1$. Therefore, $u_1, u_2, \ldots, u_{2n+1}$ form a chordless $(2n+1)$-cycle in $K(n,2n+1)$.

We only need prove that any vertex $v = \{b_1, b_2, \ldots, b_n\}$ is contained in some $(2n+1)$-cycle with $u_1 = \{1,2,3,...n\}$. If the n-subsets corresponding to u_1 and v are disjoint, then we permute $\{1,2,3,\ldots,2n+1\}$ as follows: take the product of the 2-cycles $(b_j\ (n+j))$ for each $1 \leq j \leq n$, and fix everything else. This induces an automorphism of $K(n,2n+1)$ which maps v to u_2 and fixes u_1. Hence u_1 and v are in some $(2n+1)$-cycle because u_1 and u_2 are. On the other hand, suppose u_1 and v have non-empty intersection, say $|u_1 \cap v| = t-1$ for some $2 \leq t \leq n$. Recall that $|u_1 \cap u_{2t}| = t-1$. We permute $\{1,2,3,\ldots,2n+1\}$ by interchanging the elements in $u_1 \cap v$ with those in $u_1 \cap u_{2t}$, and interchanging the elements in $v - u_1 \cap v$ with those in $u_{2t} - u_1 \cap u_{2t}$. This induces an automorphism of $K(n,2n+1)$ which maps v to u_{2t} and fixes u_1. Hence u_1 and v are in some $(2n+1)$-cycle because u_1 and u_{2t} are. □

Hence the Kneser graph $K(n,2n+1)$ is strongly $(2n+1)$-angulated graphs for each integer $n \geq 1$. For example, any two vertices of the Petersen graph are in some 5-cycle. By the Erdös-Ko-Rado Theorem, it is well-known that any two maximum independent sets of $K(n,2n+1)$ have non-empty intersection if $n \geq 2$. Hence $K(n,2n+1)\square K_2 \not\rightarrow K(n,2n+1)$ if $n \geq 2$.

Corollary 5.8. $K(n,2n+1)\square K(m,2m+1)$ is a core for any integers $m,n \geq 2$.

Proof. If $m=n$, then $K(n,2n+1)\square K(n,2n+1)$ is a core by Theorem 5.3. If $n<m$, then $K(n,2n+1)\square K(m,2m+1)$ is a core by Theorem 5.5. □

Corollary 5.9. $K(n,2n+1)\square C_{2l+1}$ is a core for any integers $l \geq n \geq 2$.

Proof. If $l=n$, then $K(n,2n+1)\square C_{2n+1}$ is a core by Theorem 5.4. If $l > n \geq 2$, then $K(n,2n+1)\square C_{2l+1}$ is a core by Theorem 5.5. □

Example 3: Although, using any 3-coloring, we see that $P\Box C_5 \to K_3$, by Corollary 5.9, $P\Box C_5$ is a core. Another proof of this fact follows from the No-homomorphism lemma (see [5]), since $P\Box C_5$ is vertex-transitive and its independence ratio is $\frac{17}{50}$. The independence ratio of any subgraph of $P\Box C_5$ must have smaller denominator, hence cannot equal $\frac{17}{50}$.

6. Questions

Question 6.1. In general, if G and H are $(2k+1)$-angulated cores, then $(G\Box H)^*$ need not be one of G, H, or $G\Box H$, see [3]. Are there extra hypotheses or G or H which will mean this must be true?

Question 6.2. If G is a strongly $(2k+1)$-angulated graph, then $G\Box G$ is not strongly $(2k+1)$-angulated, but is $(2k+1)$-angulated. What can be proved about the core of the box product of G with itself m times? For example, it is well-known that the core of the box product of C_5 with itself m times is C_5, see [8], and that the box product of P with itself m times is a core, for each m, see [1].

References

[1] M. O. Albertson and K. L. Collins, Homomorphisms of 3-chromatic number, J. Graph Theory 54 (1985), 127–132.

[2] J. A. Bondy and U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, 1976, p. 118.

[3] Z. Che, K. L. Collins, and C. Tardif, A note on $(2k+1)$-angulated graphs, preprint.

[4] C. Godsil and G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[5] G. Hahn and C. Tardif, "Graph homomorphisms: structure and symmetry", Graph Symmetry (Montreal, PQ 1996), G. Hahn and G. Sabidussi, (Editors), NATO Adv. Sci. Inst. Ser. C: Math. and Phys. Sci., 497, Kluwer Academic Publishers Group, Dordrecht, 1997, pp. 107–166.

[6] D. A. Holton and J. Sheehan, The Petersen Graph, Australian Math. Soc. Lecture Ser. 7, Cambridge Univ. Press, 1993.

[7] S. Klavžar, Two remarks on retracts of graph products, Disc. Math. 109 (1992), 155–160.

[8] B. Larose, F. Laviolette and C. Tardif, On normal Cayley graphs and hom-idempotent graphs, European J. Combin. 19 (1998), 867–881.

[9] R. Nowakowski and I. Rival, Retract rigid Cartesian products of graphs, Disc. Math. 70 (1988), 169–184.

[10] S. Stahl, n-tuple colorings and associated graphs, J. Combin. Theory Ser. (B) 20 (1976), 185–203.

[11] N. Van Ngoc and Z. Tuza, 4-chromatic graphs with large odd girth, Disc. Math. 138 (1995), 387-392.

[12] D. A. Youngs, 4-chromatic projective graphs, J. Graph Theory 21, no. 2, 219-227.

[13] E. Welzl, Symmetric graphs and intepretaions, J. Combin. Theory Ser. (B) 37 (1984), 235–244.

Department of Mathematics, Penn State Beaver, Monaca, PA 15061-2799
E-mail address: zche@wesleyan.edu, ZXC10@psu.edu

Department of Mathematics and Computer Science, Wesleyan University, Middletown, CT 06459-0128
E-mail address: kcollins@wesleyan.edu