# Most Graphs are Edge-Cordial

Karen L. Collins
Dept. of Mathematics
Wesleyan University
Middletown, CT 06457

and

Mark Hovey
Dept. of Mathematics
MIT
Cambridge, MA 02139

**Abstract**

We extend the definition of edge-cordial graphs due to Ng and Lee for graphs on $4k$, $4k+1$, and $4k+3$ vertices to include graphs on $4k+2$ vertices, and show that, in fact, all graphs without isolated vertices are edge-cordial. Ng and Lee conjectured that all trees on $4k$, $4k+1$, or $4k+3$ vertices are edge-cordial.

Intuitively speaking, a graph $G$ is said to be edge-cordial if its edges can be labelled either 0 or 1 so that half of the edges are labelled 0; half are labelled 1; and half of the vertices meet an even number of edges labelled 1, while the other half meet an odd number of edges labelled 1. This definition is due to Ng and Lee, see [2], and is the dual concept to cordial graphs, first introduced by Cahit in [1]. Ng and Lee showed that if the number of vertices in any graph is congruent to 2 (modulo 4), then the graph cannot be edge-cordial. We extend their definition of edge-cordial to graphs on $4k+2$ vertices.

**Definition 1** *Suppose we have a simple graph $G = (V, E)$. Let $f : E \to \{0, 1\}$ be an edge-labelling of $G$. Define a vertex-labelling $f^* : V \to \{0, 1\}$ by*

$$f^*(v) = \sum_{(u,v)\in E} f(u, x) \text{ (mod 2)}$$

*Let $E_i$ denote the set of edges labelled $i$, and $V_i$ be the set of vertices labelled $i$, for $i = 0, 1$. Then $G$ is said to be edge-cordial if and only if*

1. *(Ng and Lee) $| V | \equiv 0, 1, 3$ (mod 4) and there exists an edge-labelling $f$ such that $| \ (| E_0 | - | E_1 |) \ | \leq 1$ and $| \ (| V_0 | - | V_1 |) \ | \leq 1$; or*

2. $| V | \equiv 2$ *(mod 4), and there exists an edge-labelling* $f$ *such that* $| (| E_0 | - | E_1 |) | \leq 1$ *and* $| (| V_0 | - | V_1 |) | = 2$.

**Conjecture 1 (Ng and Lee)** *All trees on* $4k$, $4k+1$, *or* $4k+3$ *vertices are edge-cordial.*

Graphs with $4k+2$ vertices cannot satisfy the first condition in the definition because of the following lemma.

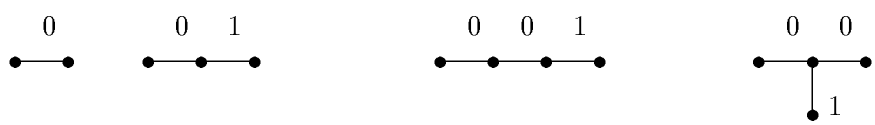
**Lemma 1 (Ng and Lee)** $| V_1 |$ *is even.*

**Proof** Consider the subgraph $H$ of $G$ induced by the edges labelled 1. The vertices labelled 1 in $G$ are the vertices of odd degree in $H$. Clearly there must be an even number of vertices of odd degree in $H$.

In this paper, we prove that if $G$ is a graph with no isolated vertices then $G$ is edge-cordial. In particular, Ng's and Lee's conjecture is true.

**Theorem 1** *Trees are edge-cordial.*

**Proof** It is easy to check that trees with 2, 3 or 4 vertices are edge-cordial.

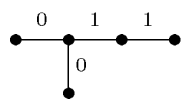



We proceed by induction. Suppose $T$ is a tree with $\geq 5$ vertices and suppose that all trees with $| V | - 4$ vertices are edge-cordial. We show that $T$ is edge-cordial.

First assume that $T$ has two leaves, $v_1$ and $v_2$, both adjacent to $w$. If $T$ has at least four leaves, we can remove $v_1$ and $v_2$ and two other leaves, $u_1$ and $u_2$, (and their respective edges) and label the remaining graph by induction. We extend the labelling to $T$ by coloring the edges incident with $v_1$ and $v_2$ both 1 and the other two edges both 0. This adds 2 vertices that see an odd number of ones, and 2 that see an even number, without changing the parity of previously labelled vertices. The vertices $u_1$ and $u_2$ may be adjacent to $w$ or not. If $T$ has less than 4 leaves, then it has either 2 or 3 leaves. If $T$ has only 2 leaves, then $T$ cannot have at least 5 vertices.

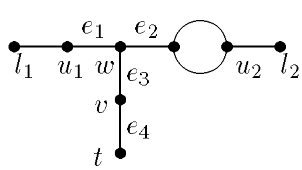
If $T$ has 3 leaves, then $T$ minus $v_1$ and $v_2$ must be a path, $P$. If $P$ has at least 4 vertices we can remove 4 of them and color them


and complete the coloring by induction. If $P$ contains $< 4$ vertices, then (since $T$ has at least 5 vertices) $T$ is

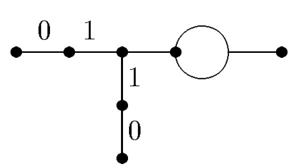



which is easily labelled as shown.

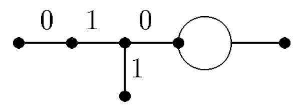
Now suppose that $T$ does not have 2 leaves both adjacent to the same vertex. Let $l_1$, $l_2$ be 2 (vertex) leaves which are maximally distant in $T$. Let $u_1$ and $u_2$, respectively, be their neighbors. We will show that the degree of $u_1$ (and hence, by symmetry, the degree of $u_2$) is 2. Obviously the degree of $u_1 \geq 2$ since $u_1$ must have an edge $e_1$ included in the path between $l_1$ and $l_2$. Suppose $u_1$ has another edge $d$ connecting $u_1$ to $v$. Then $v$ must be a leaf, since $l_1$ and $l_2$ are maximally distant. But then $l_1$ and $v$ are both leaves and both adjacent to $u_1$. So degree of $u_1 = 2$. Let $w$ be the other end of $e_1$. Let $e_2$ be the edge adjacent to $w$ but not $u_1$ in the path between $l_1$ and $l_2$. Suppose $w$ has another edge $e_3$ connecting it to a vertex $v$. Any edges from $v$ other than $e_3$ must end in leaves since $l_1$ and $l_2$ are maximally distant. Since $T$ has no two leaves both adjacent to the same vertex, there can be at most one edge from $v$ other that $e_3$, say $e_4$. Let the other end of $e_4$ be $t$. If $e_4$ exists, $T$ looks like




But then $T - \{l_1, u_1, v, t\}$ has an edge-cordial labelling, and we can complete that labelling as shown:

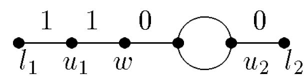


Hence we can assume that $e_4$ does not exist, and that $v$ is a leaf. There cannot be another edge besides $e_1, e_2, e_3$ coming off from $w$, since, using the same argument, we could conclude that it leads to a leaf, and we would have 2 leaves both adjacent to $w$. So $T$ looks like:




and we can complete an edge-cordial labelling as shown.

The remaining case is when the degree of $w$ is 2, and $T$ looks like




In this case we can complete an edge-cordial labelling of $T - \{l_1, u_1, w, l_2\}$ as shown. So by induction $T$ is edge-cordial.

We can use this result to prove

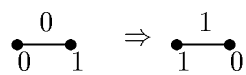
**Theorem 2** *Forests with no isolated vertices are edge-cordial.*

**Proof** Let $F$ be a forest with connected components $F_1, F_2, \ldots, F_k$. Each $F_i$ is a tree. If one of the $F_i$ has at least $\mid V_i \mid \geq 6$ vertices, we can use the same argument that we used in the induction argument for trees to reduce the number of vertices in $F_i$ to $\mid V_i \mid - 4$. Since $6 - 4 = 2$, this new tree will not be an isolated vertex. (This is because we took away the same number of 0 edges as 1 edges, and didn't change any vertex labels. Also, we left a connected graph.) Thus we need only consider forests $F$ with each $F_i$ having 2, 3, 4, or 5 vertices. If there are two 2 vertex trees, we can color one edge 1 and the other 0. Then we get two 0 vertices and two 1 vertices, so we can reduce this case. If there are two trees with 4 vertices, we can use a coloring on one with 2 zero edges and 1 one edge, and on the other, 2 one edges and 1 zero edge. One of the four cases of this is given below.

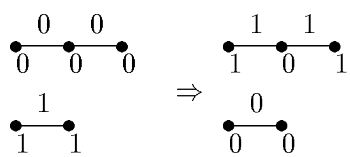
Using similar techniques (of matching a small tree with more zero edges to one with more one edges) we can reduce the following cases: a 3 vertex tree and a 5 vertex tree; four 3 vertex trees; four 5 vertex trees. Thus we have reduced to a bounded number of each kind of tree and one can just check all possible graphs.

Suppose we have an edge-cordial graph $G$ with more 0 edges than 1 edges. We would like to change the labelling so that we have more 1 edges than 0 edges. Below we list some situations in which we can do this. We will show a piece of $G$ together with the changes we make on that piece to fix the labelling.

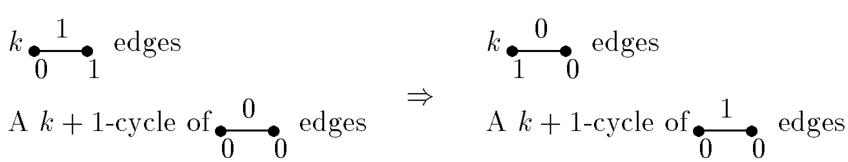
<u>Situation 1</u>



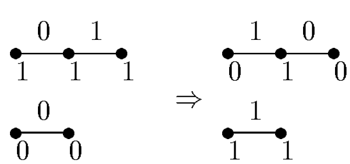
<u>Situation 2</u>



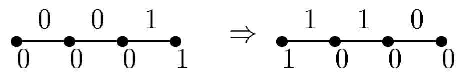
<u>Situation 3</u>



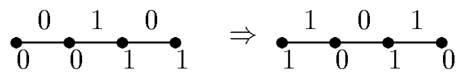

<u>Situation 4</u>



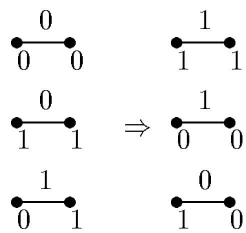
Situation 5



Situation 6



Situation 7



Each of these situations has a dual obtained by interchanging the vertex labellings. If we set $V_0 \to V_1'$ and $V_1 \to V_0'$, then we obtain a new situation which also works. Situations 1, 6, and 7 are self-dual.

**Theorem 3** *If $G$ is an edge-cordial graph with no isolated vertices and an odd number of edges, then there is an edge-cordial labelling of $G$ with*

$$| E_1 | \geq | E_0 |$$

**Proof** Choose an edge-cordial labelling of $G$. Suppose with this labelling $| E_0 | > | E_1 |$. By situation 1 above we may assume that there are no 0 edges between oppositely labelled vertices. Thus there must be either a 0 edge between two 0 vertices or a 0 edge between two 1 vertices.

**Case 1:** The only 0 edges are between two 0 vertices. If $| V | \equiv 2 \pmod 4$, we can assume $| V_0 | < | V_1 |$, since otherwise we could change a 0 edge between two 0 vertices to a 1 edge and still have an edge-cordial labelling. Now there must be two adjacent 0 edges. If not, $| E_0 | \leq | V_0 | /2$ so $| E | < | V | /2$ and $G$ would have to have an isolated vertex. Hence by Situation 2, we may assume there are no edges labelled 1 between two 1 vertices. Thus the only edges adjacent to 1 vertices are labelled 1 and connect a 1 vertex to a 0 vertex. Let $k$ denote the number of such edges.

Since $G$ has no isolated vertices $k \geq |V_1| \geq (|V| - 1)/2$. Hence by Situation 3, we may assume that there are no cycles of 0 edges of size $\leq (|V| + 1)/2$. But there are at most $(|V| + 1)/2$ vertices labelled 0. Thus the subgraph of 0 vertices and 0 edges forms a forest. So we have

$$|V_1| \leq |E_1| \leq |E_0| \leq |V_0| - 1$$

which violates the edge-cordiality of $G$.

**Case 2:** The only 0 edges are between two 1 vertices. This case is the same as the previous case. One just uses the duals of the situations used above.

**Case 3:** Both kinds of 0 edges appear. If $|V| \equiv 2 \pmod 4$, then either

$$|V_0| = |V_1| + 2 \text{ or } |V_1| = |V_0| + 2$$

In either case we can change a 0 edge to a 1 edge and still have an edge-cordial labelling. So we assume $|V| \not\equiv 2 \pmod 4$.

Since we have both kinds of 0 edges we can assume, by Situation 4 and its dual, that if a 0 edge and a 1 edge are adjacent the 1 edge must have oppositely labelled vertices. Suppose the 0 edges between 1 vertices do not form a matching. Take two adjacent ones. They form a two-edge path. The endpoints of this path are labelled 1 so they must be adjacent to a 1 edge. The only possibility is a 1 edge between oppositely labelled vertices. Thus we are in the dual of Situation 5 and we can reduce.

So we can assume that the 0 edges between 1 vertices form a matching. Suppose there is more than one such edge. Take a vertex adjacent to one of them. Again since it is labelled 1 it is also adjacent to a 1 edge between oppositely labelled vertices. If the 0 vertex it is adjacent to has a 0 edge coming off it we are in Situation 6. On the other hand if that 0 vertex does not have a 0 edge coming off it we are in Situation 7.

Hence we can assume there is only one edge labelled 0 between two 1 vertices. We know that there are two edges labelled 1, one for each endpoint of that edge. Let $l$ denote the number of other 1 edges. Let $m$ denote the number of 0 edges between two 0 vertices. Then $l + 2 = |E_1| < |E_0| = m + 1$ so $2l \leq 2m - 4$. Now if the 0 edges form a matching we have

$$|V_1| \leq 2l + 2 < 2m - 1 \leq |V_0| - 1$$

which violates edge-cordiality.

Thus the 0 edges do not form a matching and by Situation 2 we may assume there are no edges labelled 1 between 1 vertices. Since every 1 vertex must be adjacent to a 1 edge there are at least $| V_1 |$ 1 edges between oppositely labelled vertices. Thus by Situation 3, there are no cycles of size less than or equal to $| V_1 | +1$ of 0 edges between 0 vertices. Since there are at most $| V_1 | +1$ vertices labelled 0, the 0 edges form a forest. If this forest does not cover all the 0 vertices we have

$$| V_1 | \leq | E_1 | < | E_0 | \leq (| V_0 | -2) + 1 = | V_0 | -1$$

contradicting edge-cordiality. Thus every 0 vertex is adjacent to a 0 edge. Consider the 0 edge between two 1 vertices. An endpoint of this edge must have a 1 edge connecting it to a 0 vertex coming off it. This 0 vertex is connected to another 0 vetex. Thus we are in Situation 6 and we can reduce.

The last two theorems immediately imply

**Theorem 4** *Graphs with no isolated vertices are edge-cordial.*

**Proof** We proceed by induction on the number of edges attached to a spanning forest. The base case is Theorem 2. Suppose all forests with $k$ extra edges are edge-cordial. If $G$ is a forest with $k+1$ extra edges, delete an edge that is not in a spanning forest. By induction this graph is edge-cordial. By Theorem 3 we can find an edge-cordial labelling with $| E_0 | \leq | E_1 |$. Label the new edge 0. Then this labelling is edge-cordial.

# References


[1] I. Cahit, *Cordial graphs: a weaker version of graceful and harmonious graphs*, Ars. Comb. **23** 201-207 (1987).

[2] H. K. Ng and S. M. Lee, *A conjecture on edge-cordial trees*, Abstracts Amer. Math. Soc. **9** 286-287 (1988).