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HOMOMORPHISMS OF 3-CHROMATIC GRAPHS

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This paper examines the effect of a graph homomorphism upon the chromatic difference sequence of a graph. Our principal result (Theorem 2) provides necessary conditions for the existence of a homomorphism onto a prescribed target. As a consequence we note that iterated cartesian products of the Petersen graph form an infinite family of vertex transitive graphs no one of which is the homomorphic image of any other. We also prove that there is a unique minimal element in the homomorphism order of 3-chromatic graphs with non-monotonic chromatic difference sequences (Theorem 1). We include a brief guide to some recent papers on graph homomorphisms.

A homomorphism f from a graph G to a graph H is a mapping from the vertex set of G (denoted by $V(G)$) to the vertex set of H which preserves edges, i.e., if (u, v) is an edge in G, then $(f(u), f(v))$ is an edge in H. If in addition f is onto $V(H)$ and each edge in H is the image of some edge in G, then f is said to be onto and H is called a homomorphic image of G. We define the homomorphism order $\geqslant$ on the set of finite connected graphs as $G \geqslant H$ if there exists a homomorphism which maps G onto H. Graph colorings provide the most common examples of homomorphisms: an r-coloring of G is just a homomorphism to the r-clique. As a further example note that identifying the antipodal vertices of a dodecahedron is a homomorphism onto the Petersen graph. There is no homomorphism from the Petersen graph to the 5-cycle as any such mapping must identify two non adjacent vertices which necessarily produces a triangle.

Graph homomorphisms have arisen separately in three main areas. However, most investigators seem to be unaware of work outside their own area. We present several basic references. The earliest appearance of graph homomorphisms was in the categorical work of the Prague school. A typical result here is that given any monoid M, there is a graph (cubic graph, k-chromatic graph, etc.) whose endomorphism monoid is M [6]. Computer scientists who study formal languages know of graph homomorphisms in the guise of interpretations of grammars [9–13, 16]; see Salomaa's lovely monograph [13]. A typical result from this area is that given any two (homomorphism) minimal graphs each with more than one edge, then there exists a minimal graph which is naturally between the two [16]. Here the order is provided by homomorphism into. To decide whether a graph is 3-colorable is, of course, one of the classic NP-complete problems. It is not surprising that this result has been extended to more general targets. Specifically, Maurer, Sudborough, and Welzl have shown that given a fixed odd cycle it is NP-complete to decide if there exists a homomorphism from a graph G to the odd cycle [11]. They conjecture that this result holds for any homomorphism minimal target with more than one edge. Vesztergombi has shown that there exists a homomorphism from a graph G onto the 5-cycle if and only if the chromatic number of the strong product of G and the 5-cycle equals 5 [14, 15].

Given a graph G, the chromatic difference sequence of G, denoted by $\mathrm{CDS}(G) = a(1), a(2), \ldots,$ is defined for $t = 1, 2, \ldots$ by

$$\sum_{j=1}^{t} a(j) = \alpha(t, G) = \alpha(t) = \text{maximum \# of vertices in an induced } t\text{-colorable subgraph of } G.$$

This is a generalization of the antichain partition sequence for posets explored by Greene and Kleitman [4, 5]. Albertson and Berman have provided necessary and sufficient conditions for a sequence of no more than 4 terms to be the CDS of some graph [1]. As examples the CDS of the 5-cycle is 2, 2, 1 while the CDS of the Petersen graph is 4, 3, 3. There is a unique graph D shown in Fig. 1 whose CDS is 3, 1, 2.

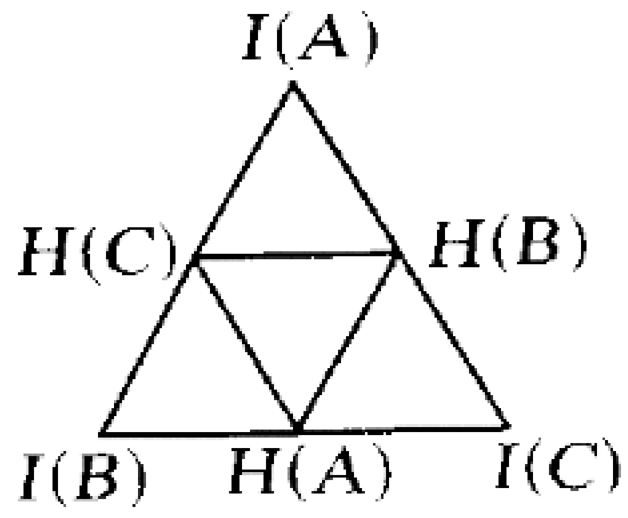

Fig. 1.

Theorem 1. *If* $\mathrm{CDS}(G) = a(1), a(2), a(3)$ *and* $a(2) < a(3)$, *then there exists a homomorphism from* G *onto* D.

Proof. Let A, B, and C be the color classes in a fixed 3-coloring of G. Fix a maximum independent set I and let $H = G - I$. Denote by $I(A)$ those vertices in I which are colored A by the fixed 3-coloring. Denote by $H(A)$ those vertices in H which are colored A by the fixed 3-coloring. Similarly define $I(B)$, $H(B)$, $I(C)$, and $H(C)$. The homomorphism f from G to D is indicated by labels assigned to the vertices of D in Fig. 1. Since I is independent and A, B, and C are color classes, $f(G)$ is a subgraph of D. It remains to show that f is a homomorphism onto D. If H were 2-colorable, then a largest color class of H together with I would provide a 2-colorable subgraph of G which would be large enough to prove that $a(2) \geq a(3)$. Thus H is 3-chromatic and each of $H(A)$, $H(B)$, and $H(C)$ is not empty. Further the edges in D joining these three vertices must be images of edges in G or H would be 2-colorable. Next we show that $I(C)$ is not empty. If

$I(C)$ were empty, since I is a maximum independent set

$$a(1)=|I(A)|+|I(B)|.$$

Since $a(1)+a(2)\geq|I(A)|+|I(B)|+|H(A)|+|H(B)|$,

$$a(2)\geq|H(A)|+|H(B)|.$$

Since $a(1)+a(2)\geq|I(A)|+|I(B)|+|H(C)|$,

$$a(2)\geq|H(C)|.$$

But

$$|H(A)|+|H(B)|+|H(C)|=a(2)+a(3)$$
$$>2a(2)\geq|H(A)|+|H(B)|+|H(C)|.$$

Similarly $I(B)$ and $I(A)$ are not empty. Finally we need to show that every edge in D is the image of some edge in G. We will actually show more. Suppose that $I(C)$ does not contain a vertex which is joined to both some vertex in $H(A)$ and some vertex in $H(B)$. Let $I(C, A)$ denote those vertices in $I(C)$ which are adjacent to no vertex in $H(A)$. Similarly define $I(C, B)$. Clearly $I(C)$ is the union of $I(C, A)$ and $I(C, B)$. Both $I(A)\cup H(A)\cup I(C, A)$ and $I(B)\cup H(B)\cup I(C, B)$ form independent sets of vertices. Thus

$$a(1)+a(2)\geq|V(G)|-|H(C)|.$$

Consequently $a(3)\leq|H(C)|$.

Since I and $H(C)$ together form a 2-colorable subgraph of G,

$$a(2)\geq|H(C)|.$$

Thus if D is not a homomorphic image of G, then $a(2)\geq a(3)$. □

Suppose there exists a homomorphism from G to H. By definition, the inverse image of an independent set in H is independent in G. Thus any coloring of H pulls back to a coloring of G. One can interpret this remark as saying that a necessary condition for the existence of a homorphism from G to H is that the chromatic number of H must be at least as large as the chromatic number of G. Our principal result is an extension of this necessary condition to chromatic difference sequences. For convenience we define the normalized chromatic difference sequence of a graph G, denoted NCDS(G). If CDS(G)= $a(1), a(2), \ldots,$ then NCDS(G)$=c(1), c(2), \ldots$ where $c(j)$ is defined to be $a(j)/|V(G)|$. The purpose of this is to allow comparisons using the dominance relation. Given two sequences $b(1), b(2), \ldots, b(r)$ and $d(1), d(2), \ldots, d(r)$ the sequence of b's is said to dominate the sequence of d's if

$$\sum_{j=1}^{t} b(j)\geq\sum_{j=1}^{t} d(j)$$

for $t=1, 2, \ldots, (r-1)$ and equality holds for $t=r$. We recommend Marshall and Olkin's treatise to the interested reader [8].

Theorem 2 (The no-homomorphism lemma). *If there exists a homomorphism from G to H and H is vertex transitive, then* NCDS(G) *dominates* NCDS(H).

Proof. We begin by simplifying some of the notation introduced above. Let $\alpha(G)$ denote the maximum number of vertices in a t-colorable induced subgraph of G. To show that NCDS(G) dominates NCDS(H), we need to show that

$$\alpha(G)/|V(G)| \geqslant \alpha(H)/|V(H)|.$$

Suppose $I(1), I(2), \ldots, I(p)$ are all of the maximum t-colorable induced subgraphs of H. Denote by m the number of these subgraphs which contain a fixed vertex x. Since H is vertex transitive m is well defined. We can express $\alpha(H)/|V(H)|$ explicitly:

$$p\alpha(H) = \sum_{j=1}^{p} |I(j)| = \sum_{j=1}^{p} \sum_{x \text{ in } V(H)} |x \cap I(j)|$$
$$= \sum_{x \text{ in } V(H)} \sum_{j=1}^{p} |x \cap I(j)| = |V(H)|\, m.$$

Thus $\alpha(H)/|V(H)| = m/p$. Next we perform a similar trick on G. Let $L(j)$ denote the inverse image of $I(j)$ under the homomorphism f. As previously noted since $I(j)$ is an induced t-colorable subgraph of H, $L(j)$ is an induced t-colorable subgraph of G. Note it is not necessarily the case that $L(j)$ is a maximum t-colorable induced subgraph of G, nor is it the case that there are exactly p such subgraphs of G.

$$p\alpha(G) \geqslant \sum_{j=1}^{p} |L(j)| = \sum_{j=1}^{p} \sum_{x \text{ in } V(G)} |x \cap L(j)|$$
$$= \sum_{j=1}^{p} \sum_{x \text{ in } V(H)} |f^{-1}(x) \cap L(j)| = \sum_{j=1}^{p} \sum_{x \text{ in } V(H)} |f^{-1}(x)|\, |x \cap I(j)|$$
$$= \sum_{x \text{ in } V(H)} \sum_{j=1}^{p} |f^{-1}(x)|\, |x \cap I(j)| = \sum_{x \text{ in } V(H)} |f^{-1}(x)| \sum_{j=1}^{p} |x \cap I(j)|$$
$$= |V(G)|\, m.$$

Hence $\alpha(G)/|V(G)| \geqslant m/p = \alpha(H)/|V(H)|$. $\square$

Here is an example of how the no-homomorphism lemma can work. There is no homomorphism from the Petersen graph to the 5-cycle since the NCDS of the 5-cycle is 0.4, 0.4, 0.2 and the NCDS of the Petersen graph is 0.4, 0.3, 0.3. To see that vertex transitivity is necessary let G be the 5-cycle and H be the triangle with a pendant vertex. Clearly G folds to H (a fold is just a special type of homomorphism; see [3]), yet the dominance relation doesn't hold.

Remark. The proof of the no-homomorphism lemma is sufficiently general to apply to other invariants of the graph. As an example let $\delta(t, G)$ denote the maximum number of vertices in an induced subgraph of G which contains no t-clique. Since a t-colorable subgraph of G necessarily contains no $(t+1)$-clique, $\delta(t+1, G) \geq \alpha(t, G)$. It can be shown, analogously to the above, that if there exists a homomorphism from a graph G onto a vertex transitive graph H, then

$$\delta(t, G)/|V(G)| \geq \delta(t, H)/|V(H)|.$$

We next use the no-homomorphism lemma to show the existence of an infinite family of 3-chromatic graphs no one of which is the homomorphic image of any other. Given graphs H and K the cartesian product of H and K, denoted by $H \times K$ has $V(H \times K) = V(H) \times V(K)$. Edges in the product are given by $(u, v) \sim (x, y)$ if either $u = x$ and $v \sim y$ or $u \sim x$ and $v = y$. Let $P(1)$ denote the Petersen graph. Set $P(n)$ equal to the cartesian product of $P(1)$ with $P(n-1)$.

Theorem 3. *For any $r > s$ there does not exist a homomorphism from $P(r)$ onto $P(s)$.*

Proof. Since the Petersen graph is vertex transitive so is $P(s)$. It suffices to show that the NCDS($P(r)$) does not dominate the NCDS($P(s)$). This follows immediately from the claim that

$$\text{NCDS}(P(r)) = 1/3((1+2/10 ** r), (1-1/10 ** r), (1-1/10 ** r)).$$

This is equivalent to the claim that CDS($P(r)$) $= a(1)$, $a(2)$, $a(3)$ where $a(1) = (10 ** r+2)/3$ and $a(2) = (10 ** r-1)/3 = a(3)$. The proof of this consists of two parts. First we produce a 3-coloring of $P(r)$ which shows that CDS($P(r)$) dominates $a(1)$, $a(2)$, $a(3)$. Assume that we have 3-colorings of $P(1)$ and $P(r-1)$ fixed. If $x = (u, v)$ is a vertex in $P(r)$ set the color of x equal to the color of u plus the color of v (mod 3). It is straightforward to check that adjacent vertices in $P(r)$ are assigned different colors. Next we must show that the largest independent set in $P(r)$ cannot contain more than $a(1)$ vertices. A maximum independent set in $P(r)$ can be thought of as 10 independent sets in $P(r-1)$. We use induction to note that the largest independent set in $P(r-1)$ has no more than $(10 ** (r-1)+2)/3$ vertices. There can be no more than four copies of $P(r-1)$ which have independent sets this large. The remaining six copies of $P(r-1)$ can have independent sets of size no more than $(10 ** (r-1)-1)/3$. Thus the independence number of $P(r)$ is no more than $a(1)$. The proof that the largest 2-colorable subgraph of $P(r)$ contains no more than $a(1)+a(2)$ vertices is similar. □

We obtain a minor strengthening of the above theorem by noting that since the Petersen graph is rigid, i.e., it has no endomorphisms which are not automorphisms, so is $P(s)$. Consequently there cannot exist a homomorphism from $P(r)$ into $P(s)$.

Sometimes a cartesian product of graphs will map to one of its factors. For example the cartesian product of two 5-cycles maps to the 5-cycle. Infinite families of 3-chromatic graphs which are incomparable in the homomorphism order have been constructed [7]. However, previous constructions are not so easy and elegant and do not produce transitive graphs.

Theorem 2 inspired us to investigate the NCDS' of vertex transitive graphs. We offer the following:

Conjecture. If G is vertex transitive, then NCDS(G) is monotonic.

This conjecture has not been verified even for the circulants. (See [2] for a summary of what is known about these graphs.) One might suspect that Theorem 1 would provide a method of proof of this conjecture for 3-chromatic graphs. However since D is eulerian there exists a homomorphism from the 9-cycle onto D.

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