Planar Lattices are Lexicographically Shellable

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Abstract

The special properties of planar posets have been studied, particularly in the 1970's by I. Rival and others. More recently, the connection between posets, their corresponding polynomial rings and corresponding simplicial complexes has been studied by R. Stanley and others. This paper, using work of A. Björner, provides a connection between the two bodies of work, by characterizing when planar posets are Cohen-Macaulay. Planar posets are lattices when they contain a greatest and a least element. We show that a finite planar lattice is lexicographically shellable and therefore Cohen-Macaulay iff it is rank-connected.

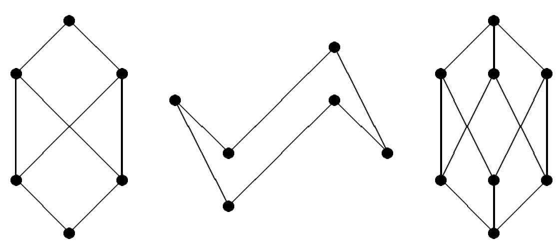
Every poset corresponds to a simplicial complex, called the order complex, defined by taking chains of the poset to be faces of the simplicial complex. We define a face of a simplicial complex to include both interior and boundary. A pure, finite simplicial complex is **shellable** if its maximal faces (facets) can be ordered $F_1, F_2, \ldots, F_n$ in such a way that $F_k \cap (\cup_{i=1}^{k-1} F_i)$ is a nonempty union of maximal proper faces of F_k for $2 \leq k \leq n$. Shellable triangulations of spheres and balls are discussed in [7]. A poset is said to be shellable if its order complex is shellable.

Let k be a field. Then with every poset P there is also associated a polynomial ring $k(P)$ whose variables correspond to the vertices of the poset, in which a product of two or more variables is zero if and only if it contains an independent set of vertices of the poset. Thus it is possible to ask if this ring is Cohen-Macaulay (C-M); *i.e.,* if the depth of $k(P)$ is equal to the (Krull) dimension of $k(P)$. See [19] for background. (The Krull dimension of a polynomial ring is different from the combinatorial definition of the dimension of a poset. See definition below.)

Let f be a face in a simplicial complex C, and $\mathbf{link}(f)$ be the subcomplex of C consisting of the set of faces g in C such that $f \cup g$ is a face in C and $f \cap g = \emptyset$. The topological dimension of a simplicial complex is defined to be one less than the size of its largest face. Then a simplicial complex is defined to be C-M when for every face f in the complex C the reduced homology of link(f) is zero except in the term $\tilde{H}_n(f;k)$, where n is the topological dimension of link(f). Reisner's theorem states that the polynomial ring of a poset is C-M iff the simplicial complex of the poset is C-M [15]. See [19] for a introduction to the combinatorics of order complexes and their homology.

In a poset P, an element y **covers** x if $y > x$ and there is no z such that $y > z > x$. The **diagram** of a poset is a drawing of the vertices and covering relations of the poset such that if y covers x, then y appears above x in the diagram, and there is an edge drawn from x to y. A poset is said to be **planar** if its diagram can be drawn in the plane with straight edges and without crossings. We say that a poset with a unique greatest element and a unique least element is a **bounded** poset. We call the greatest element $\hat{1}$ and the least element $\hat{0}$. A finite, bounded, planar poset is a lattice; see [12].

The **combinatorial dimension** of a poset P is defined to be the least d such that P is the intersection of d linear orders; see [13]. A finite, bounded, planar poset (which is a lattice) has dimension ≤ 2, see [13] for a description of the proof in [1]. The $\hat{0}$ and $\hat{1}$ of the poset guarantee that every pair of elements will have a meet and a join, while uniqueness of meet and join holds because there can be no edge crossings. There are planar posets without $\hat{0}$ and $\hat{1}$ of arbitrary dimension (see [11]) that would not be planar with $\hat{0}$ and $\hat{1}$ element added; see Figure 1 (b) and (c). Figure 1 (a) is not planar, Figure 1 (b) is the planar 6 cycle, and Figure 1 (c) is the 6 cycle with $\hat{0}$ and $\hat{1}$ added, which is the non-planar Boolean lattice on 3 elements.



(a) (b) (c)

Figure 1

A **join-(meet-)irreducible** element in a lattice covers (is covered by) exactly one element.

A finite, planar lattice always contains a **doubly irreducible** element, *i.e.* an element that is both meet- and join-irreducible. In any planar embedding of the diagram of a finite, planar lattice, we can always find a doubly irreducible element on the outside boundary of the diagram see [2]. This fact implies that finite planar lattices are dismantlable; see [16].

We can compare the diagrams of posets to planar graphs, by Platt's result that a finite, bounded poset is planar if and only if its diagram plus an edge between $\hat{0}$ and $\hat{1}$ is a planar graph [14]. For V the number of vertices in a graph, there is a $O(V)$ time algorithm to check planarity of a graph by Hopcroft and Tarjan (see [10]). Thus, it is easy to distinguish, given a poset and its diagram, whether or not it is a planar poset.

A poset P is **ranked** if for every $x \leq y$ in P, every maximal chain from x to y has the same length. A finite, bounded poset is ranked if every maximal chain from $\hat{0}$ to $\hat{1}$ has the same length. A finite, bounded poset which has every chain from $\hat{0}$ to $\hat{1}$ the same length (*i.e.* is ranked) is said to be **graded**. The diagram of a graded, planar poset P, considered as a graph, is a leveled planar graph; see [9]. A poset P is **rank-connected** if it is ranked, and every pair of consecutive ranks, considered as a vertex-induced subgraph is connected. Björner ([4]) invented the concept of lexicographic shellability. With this concept, he proved a conjecture of Stanley, that all supersolvable lattices are C-M. In fact he proves that lexicographically shellable posets are shellable, and all finite upper semi-modular lattices are lexicographically shellable. Shellable posets are C-M; see [18].

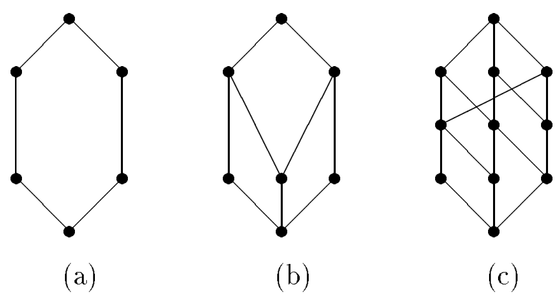


Figure 2

Thus, Figure 1 (c) is distributive (and modular) hence lexicographically shellable, while Figure 2 (a) is planar, but not rank-connected, and therefore not lexicographically shellable. Figure 2 (b) is planar, rank-connected and lexicographically shellable, but not upper semi-modular. The diagram of Figure 2 (c) itself is not planar, but considered as a graph is planar. It is rank-connected, but not lexicographically shellable. Figure 1 (a) is neither a lattice, nor is it planar, but it is lexicographically shellable.

Let P be a graded poset. Let $C(P)$ equal the set of covering relations of P. Then P is said to be **lexicographically shellable** if there exists an $f : C(P) \to \mathbf{R}$ which is an edge labeling of the diagram with labels from the real numbers that satisfies:

1. in every interval $[x, y]$ of P there is a unique unrefinable chain $x = x_0 < x_1 < \ldots < x_n = y$ such that $f(x_0, x_1) \leq f(x_1, x_2) \leq \ldots \leq f(x_{n-1}, x_n)$.
2. for every interval $[x, y]$ of P, if $x = x_0 < x_1 < \ldots < x_n = y$ is the unique unrefinable chain with rising labels, and if $z \in [x, y]$ covers x with $z \neq x_1$, then $f(x, x_1) < f(x, z)$.

In other words, P is lexicographically shellable if in any interval, the first edge of the unique rising chain of that interval has a label strictly smaller than all of the labels of the other edges in the interval that cover the smallest element of the interval. Such an edge labeling f is called an EL-labelling.

Define a **corner** to be a doubly irreducible element v that is covered by y, covers x, and has an opposite $w \neq v$ that is covered by y and covers x. We show that if a corner is added to any lexicographically shellable poset then the resulting poset is lexicographically shellable.

Theorem 1 *Let P be a lexicographically shellable poset. Let $x < w < y$ be covering relations. Then if we add the corner v, with $x < v < y$ as covering relations, the poset $P + v$ is lexicographically shellable.*

Define $f : C(P) \to \mathbf{R}$ as an EL-labeling of P. We extend f to include v by labeling the edge between x and v as a real number greater than $f(s, x)$ for every s that x covers or is covered by. We label the edge between v and y as a real number less than $f(x, v)$ and $f(y, t)$ for every cover t of y. This labeling clearly satisfies the two conditions of lexicographic shellability in the interval $[x, y]$. In fact, this is an EL-labeling of $P + v$. For any interval $[s, t]$ in $P + v$, we have v in $[s, t]$ if only if $s \leq x$ and $y \leq t$ or $s = v$ or $v = t$. If v is not in $[s, t]$ the unique rising chain remains the same. If v is in $[s, t]$, and $[x, y] \subseteq [s, t]$, then v is not part of any rising chain, since the two edges (x, v) and (v, y) form a falling chain, and any chain containing v must contain these two edges. If $s = v$, then the first edge in any chain from v to t must be (v, y), and the weight of this edge is less than all covers of y, and so the unique rising chain in $[y, t]$ is still rising when we add (v, y). The same idea works if $t = v$.

Theorem 2 *Let P be a graded, planar, rank-connected poset. Then P is lexicographically shellable.*

PROOF Baker, Fishburn and Roberts [2] have shown that a planar poset with at least 3 vertices has at least one doubly irreducible element. First we shall show that such an element is a corner. Then we proceed by induction.

Let P have n vertices and the ranks of P be labelled $R_0, R_1, \ldots, R_{s+1}$ with $R_0 = \{\hat{0}\}$, $R_{s+1} = \{\hat{1}\}$, and the labels in between in consecutive order. Embed the diagram G of P plus the edge ϵ between $\hat{0}$ and $\hat{1}$ in the plane. Then erase ϵ. We can assume that $\hat{0}$ and $\hat{1}$ are in the same face of the embedding.

Suppose that one of the R_i, $1 \leq i \leq s$, has cardinality equal to 1. Then we can divide P into $P_1 = R_i \cup R_{i+1} \cup \ldots \cup R_{s+1}$ and $P_2 = R_0 \cup R_1 \cup \ldots \cup R_i$. Now all the vertices of P_1 are greater than or equal to those in P_2. Furthermore, if we have a lexicographical shelling of P_1 using real numbers greater than a and for P_2 using real numbers less than a, then we easily obtain one for P by using the same labellings. Therefore we can assume that no R_i has cardinality equal to 1.

Lemma 1 *Let C be a cycle in G with vertex set $V(C) \subseteq R_0 \cup R_1 \cup R_2 \ldots \cup R_j$. If vertex z in P has rank greater than or equal to $j+1$, then z and $\hat{1}$ are both inside of C or both outside of C.*

PROOF (of lemma) Let $i(z, \hat{1})$ be a strictly rank increasing path from z to $\hat{1}$. Then $i(z, \hat{1})$ cannot cross C.

Lemma 2 *Let P be graded, planar, and rank-connected. Then P has a corner.*

PROOF (of lemma) Let $v \in R_j$ be a doubly irreducible element on the boundary of G, covered by y and covering x. We know v exists by [2]. Suppose that R_j contains at least two elements. Let $G_{j,j+1} = R_j \cup R_{j+1}$ for any $0 \leq j, j+1 \leq s+1$.

Let $z \neq v$ be in R_j. Since $G_{j,j+1}$ forms a connected graph, there is a shortest path $S_{j,j+1}$ from v to z in $G_{j,j+1}$. $S_{j,j+1}$ must contain y since it is the only neighbor of v in $G_{j,j+1}$. Let $w \neq v$ be the vertex adjacent to y in $S_{j,j+1}$. If w is adjacent to x, then v is a corner and w is an opposite corner to v.

Otherwise, let $C_{j-1,j}$ be the shortest path from x to w in $G_{j-1,j}$ and let z be any vertex in $R_j \cap C_{j-1,j}$. If z is adjacent to y, then z is an opposite corner to v.

Assume that no z is adjacent to y. We show a contradiction. Then $\{x, y, v, w\} \cup \{(x, v), (v, y), (w, y)\} \cup C_{j-1,j}$ forms an even chordless cycle, which we call C_1. Let $i(s,t)$ be the rank increasing path from s to t. Then $i(\hat{0},x) \cup i(\hat{0},w) \cup C_{j-1,j}$ forms a cycle, or a cycle with an extra edge with end the vertex w if the neighbor of w in $i(\hat{0},w)$ is the same as the neighbor of w in $C_{j-1,j}$. Call this cycle C_2. Then consider where $\hat{1}$ can be; either inside of C_1, C_2 or outside of both. If $\hat{1}$ is inside of C_1, then we cannot add the edge ϵ between $\hat{0}$ and $\hat{1}$. If $\hat{1}$ is inside of C_2, then y and $\hat{1}$ are separated by C_2, a contradiction by Lemma 1. If $\hat{1}$ is outside of both C_1, C_2, then each $z \in R_j \cap C_{j-1,j}$ is separated from $\hat{1}$, a contradiction by Lemma 1 and the fact that no z is adjacent to y. Therefore v is a corner of P with opposite corner w.

We finish the proof of Theorem 2 by induction on the number of vertices. Clearly any chain is lexicographically shellable. Assume that every graded, planar, rank-connected poset with $\hat{0}$ and $\hat{1}$ and n or fewer vertices is lexicographically shellable. Let P be a graded, planar, rank-connected poset with $\hat{0}$ and $\hat{1}$ and $n+1$ vertices. Then P has a corner v by Lemma 2. Then $P - v$ is still graded, planar, and rank-connected. Therefore, by induction, $P - v$ is lexicographically shellable. By Theorem 1, P is also lexicographically shellable.

The proof of Theorem 1 gives a polynomial time algorithm for finding an EL-labeling of a planar lattice.

Lemma 3 *Let $\tilde{P}$ be a poset with a face f such that the diagram of link(f) is disconnected and the topological dimension of link(f) is at least 1. Then $\tilde{P}$ is not C-M.*

PROOF (of lemma) The zeroth component of the reduced homology of link(f) is not zero, since link(f) has at least two components. Moreover, the topological dimension of link(f) is greater than zero. By Reisner's theorem, $\tilde{P}$ is not C-M (and hence not lexicographically shellable).

Theorem 3 *A graded poset P that is not rank-connected is not lexicographically shellable.*

PROOF (of theorem) Let the rank of $\hat{0}$ be 0 and the rank of $\hat{1}$ be d. Let S be a subset of the ranks of P that always includes 0 and d. Let P_S be the S-rank-selected subposet of P. It is shown in [6] (Theorem 5.2) that if P is C-M, then so is P_S. Let i and $i+1$ be the consecutive ranks in P which do not form a connected graph, and let $S = \{0, i, i+1, d\}$. We show that P_S is not C-M by Lemma 3 above. Choose $f = \{\hat{0}, \hat{1}\}$. Then link(f) is exactly the bipartite graph between ranks i and $i+1$, which is not connected.

Let $\hat{P}$ be P with a $\hat{0}$ and $\hat{1}$ added.

Lemma 4 *P is C-M if and only if $\hat{P}$ is C-M.*

PROOF (of lemma) Suppose that P is C-M. Both $\hat{0}$ and $\hat{1}$ are non-zero divisors in $k(\hat{P})$. They add two to the algebraic dimension, and depth of $k(P)$ without changing anything else. Conversely, suppose $\hat{P}$ is C-M. Let f be a face in P. Then link(f) in P is equal to link($f \cup \{\hat{0}, \hat{1}\}$) in $\hat{P}$. Therefore, the reduced homology of link(f) in P is zero except for possibly in its topological dimension.

Lemma 5 *If finite, planar poset P is ranked but not bounded, then P is C-M if and only if $\hat{P}$ is rank-connected.*

PROOF (of lemma) If $\hat{P}$ is rank-connected, then by Theorem 2, $\hat{P}$ is lexicographically shellable and C-M, hence P is C-M. If P is C-M, then $\hat{P}$ is C-M. If $\hat{P}$ were not rank-connected, then $\hat{P}$ would not be C-M by Theorem 3.

We give an elementary proof of the following well-known result. See for instance, the proof of Corollary 4.2 in [19].

Theorem 4 *If P is not ranked, then P is not C-M.*

PROOF By Lemma 6 we can assume that P is bounded. Suppose that P is not ranked, and let $[x, y]$ be the interval in P with the smallest length maximal chain that also has at least two maximal chains of different lengths, say l_1 and l_2. There can be no edge from a vertex in a chain of length l_1 to a vertex in a chain of length l_2. If there were, we would have an interval with a smaller length maximal chain with at least two different sizes of maximal chain. Now since there can be no edges between chains of different lengths, the subposet $Q = [x, y] - \{x, y\}$ must be a disconnected diagram. Clearly the topological dimension of Q is at least 1. By Lemma 3, P is not C-M.

Theorem 5 *Let P be a finite, planar poset. Then P is C-M if and only if $\hat{P}$ is rank-connected.*

PROOF If P is C-M, then $\hat{P}$ is C-M by Lemma 5, and hence rank-connected by Theorem 3. Conversely, if $\hat{P}$ is rank-connected, then $\hat{P}$ is lexicographically shellable by Theorem 2, hence C-M, hence P is C-M by Lemma 5.

A further refinement of C-M lattices are admissible lattices. Admissible lattices are lexicographically shellable, but not all lexicographically shellable lattices are admissible. See Stanley's paper for details [17]. We define admissible lattices and make a conjecture. Let J be the set of join-irreducibles of a lattice. Define a natural labeling ω of J to be a map $\omega : J \rightarrow \mathbf{N}$ where $\mathbf{N}$ is the positive integers such that if $z, w \in J$ and $z \leq w$, then $\omega(z) \leq \omega(w)$. Let γ be derived from ω by $\gamma(x < y) = min\{\omega(z) | z \in J, x < x \vee z \leq y\}$. A lattice L is admissible if whenever $x < y$ in L, there is a unique unrefinable chain $x = x_0 < x_1 < \ldots < x_m = y$ such that $\gamma(x_0, x_1) \leq \gamma(x_1, x_2) \leq \ldots \leq \gamma(x_{m-1}, x_m)$.

Conjecture 1 *A poset P which is graded, rank-connected, and planar is an admissible lattice.*

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