A Bijective Proof for the Parity of Stirling Numbers

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Abstract
We give a bijective proof for the identity $S(n, k) \equiv \binom{n-j-1}{n-k} \pmod{2}$
where $j = \left\lfloor \frac{n}{k} \right\rfloor$ is the largest integer $\leq \frac{n}{k}$.

In [1], page 46, problem 17b, Richard Stanley asks for a combinatorial proof of the identity

$$S(n, k) \equiv \binom{n-j-1}{n-k} \pmod{2}$$

Here $j = \left\lfloor \frac{n}{k} \right\rfloor$ is the largest integer $\leq \frac{n}{k}$. It is the purpose of this note to provide such a proof. Recently, Sagan [2] has found a different bijection for the $q$-analogue of $S(n, k)$.

For $n$ a positive integer, let $[n]$ denote the set $\{1, \ldots, n\}$. Let $P_{n,k}$ denote the set of partitions of $[n]$ into $k$ parts, so that the cardinality of $P_{n,k}$ is $|P_{n,k}| = S(n, k)$. We are going to define involutions $f_{n,k} : P_{n,k} \rightarrow P_{n,k}$ by induction on $n$ and $k$. Clearly, $S(n, k)$ will have the same parity as the number of fixed points of the involutions $f_{n,k}$.

Let $f_{n,n}$ and $f_{n,1}$ be the identity mapping for all $n$. Now suppose we have an element in $P_{n,k}$, say $\pi = \{B_1, \ldots, B_k\}$. Suppose the number $n$ is in block $B_r$. We define $s = \max([n] - B_r)$. Let the block that $s$ is in be $B_i$. By definition, $i \neq r$. Clearly the numbers $s + 1, s + 2, \ldots, n - 1, n$ are all in $B_r$, since $s$ is the biggest number not in $B_r$. Our idea is to switch $s$ with the set of numbers $s + 1$ to $n$ to get a new partition, that is, let

$$f_{n,k}(\pi) = \{B'_1, \ldots, B'_k\}$$

where

$$B'_l = \begin{cases} 
B_l & \text{if } l \neq i, r \\
(B_l - \{s\}) \cup ([n] - \{s\}) & \text{if } l = i \\
(B_r - ([n] - \{s\}) \cup \{s\}) & \text{if } l = r 
\end{cases}$$

However this won’t work when $s$ is the only element of $B_i$ and $B_r$ is exactly the numbers from $s + 1$ to $n$, since we will simply interchange the two blocks
without altering them. In this case, we will forget about the numbers from \( s \) to \( n \) and work on the smaller set \([s - 1]\). Therefore, let

\[
f_{n,k}(\pi) = f_{n-1,k-2}(\pi - \{B_i, B_r\}) \cup \{B_i, B_r\}
\]

Then \( f_{n,k} \) is clearly an involution.

As an example, suppose that \( \pi = \{\{6, 5\}, \{4\}, \{3, 1\}, \{2\}\} \). Then

\[
f(\pi) = \{\{6, 5\}, \{4\}, \{3\}, \{2, 1\}\}
\]

Now we will count the number of partitions fixed under \( f_{n,k} \). Suppose \( f_{n,k}(\pi) = \pi \). If \( k \) is even, \( \pi \) must look like

\[
\{[n] - [s_1], \{s_1\}, [s_1 - 1] - [s_2], \{s_2\}, [s_2 - 1] - [s_3], \ldots, [s_{j-1} - 1] - [s_j], \{s_j\}\}
\]

where \( n > s_1 > s_2 > \ldots > s_j = 1 \) and \( s_i - 1 > s_{i+1} \). We have that \( s_j = 1 \) and so \( s_{j-1} \geq 3 \). Hence the number of such \( \pi \) is equal to the number of ways of choosing \( j - 1 \) non-consecutive dots out of \( n - 3 \) dots in a row.

If \( k \) is odd, then \( \pi \) must look like

\[
\pi = \{[n] - [s_1], \{s_1\}, [s_1 - 1] - [s_2], \{s_2\}, [s_2 - 1] - [s_3], \ldots, [s_{j-1} - 1] - [s_j], \{s_j\}, [s_j - 1]\}
\]

where \( n > s_1 > s_2 > \ldots > s_j > 1 \) and \( s_i - 1 > s_{i+1} \). We have that \( s_j > 1 \) and hence the number of such \( \pi \) is the number of ways of choosing \( j \) non-consecutive dots out of \( n - 2 \) dots in a row.

Let \( g(m, t) \) be the number of ways of choosing \( t \) non-consecutive dots from \( m \) dots in a row. It is well-known and easy to show that

\[
g(m, t) = \binom{m-t+1}{t}
\]

Hence, if \( k \) is even, \( S(n, k) \equiv \binom{n-3-(j-1)+1}{j-1} \pmod{2} \), and this equals \( \binom{n-j-1}{n-k} \).

If \( k \) is odd, \( S(n, k) \equiv \binom{n-2-j+1}{j} \pmod{2} \), and this is equal to \( \binom{n-j}{n-k} \).

References
